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# Conditional symmetries and Riemann invariants for hyperbolic systems of PDEs

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## Abstract

This paper contains an analysis of rank- $k$  solutions in terms of Riemann invariants, obtained from interrelations between two concepts, that of the symmetry reduction method and of the generalized method of characteristics for first order quasilinear hyperbolic systems of PDEs in many dimensions. A variant of the conditional symmetry method for obtaining this type of solutions is proposed. A Lie module of vector fields, which are symmetries of an overdetermined system defined by the initial system of equations and certain first order differential constraints, is constructed. It is shown that this overdetermined system admits rank- $k$  solutions expressible in terms of Riemann invariants. Finally, examples of applications of the proposed approach to the fluid dynamics equations in  $(k + 1)$  dimensions are discussed in detail. Several new soliton-like solutions (among them kinks, bumps and multiple wave solutions) have been obtained. These solutions remain bounded even when the Riemann invariants admit a gradient catastrophe. Some physical interpretation of these results is discussed.

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## 1. Introduction

The general properties of nonlinear systems of PDEs in many dimensions and techniques for obtaining their exact solutions remain essential subjects of investigation in modern mathematics. In the case of hyperbolic systems, the oldest, and still useful, approach to this subject has been the method of characteristics which originated from the work of G Monge [32]. In its modern form it is described, e.g. in [11, 26, 30, 40, 41, 44]. More recently, the

development of group theoretical methods, based on the work of S Lie [29], has led to progress in this area, delivering new efficient techniques. However, these two theoretical approaches have remained disconnected and have provided, in most cases, different sets of solutions. The symmetry reduction methods (SRM) certainly have a broader range of application, while the generalized method of characteristics (GMC), though limited to nonelliptic systems, has been more successful in producing wave and multiple wave solutions. Thus, the mutual relation between these two methods is a matter of interest and we have undertaken this subject with the view of combining the strengths of both of them.

The approach to constructing rank- $k$  solutions presented in this paper evolved from our earlier work [22, 24, 25], aimed at obtaining Riemann  $k$ -waves by means of the conditional symmetry method (CSM). The main idea here has been to select the supplementary differential constraints (DCs), employed by this method, in such a way that they ensure the existence of solutions expressible in terms of Riemann invariants. Interestingly, as we show later, these constraints prove to be less restrictive than the conditions required by the GMC. As a result, we obtain larger classes of solutions than the class of Riemann  $k$ -waves obtainable through the GMC.

The organization of this paper is as follows. Section 2 gives a brief account of the generalized method of characteristics for first order quasilinear hyperbolic systems of PDEs in many independent and dependent variables. A geometric formulation of the Riemann  $k$ -wave problem is presented there. In section 3 we reformulate this problem or rather, more generally, a problem of rank- $k$  solutions expressible in terms of Riemann invariants, in the language of group theoretical approach. The necessary and sufficient conditions for obtaining this type of solutions are determined after an analysis of their group properties. A new version of the conditional symmetry method for construction of these solutions is proposed. Sections 4–7 present an application of the developed approach to the equations describing an ideal nonstationary isentropic compressible fluid. We find rank-1 as well as rank-2 and rank-3 solutions admitted by the system, among them several new types of soliton-like solutions including kinks, bumps and snoidal waves. In section 8, we construct rank- $k$  solutions for the isentropic flow with sound velocity dependent on time only. We show that the general integral of a Cauchy problem for this system depends on  $k$  arbitrary functions of  $k$  variables. Section 9 summarizes the obtained results and contains some suggestions regarding further developments.

## 2. The generalized method of characteristics

The generalized method of characteristics has been designed for the purpose of solving quasilinear hyperbolic systems of first order PDEs in many dimensions. This approach enables us to construct and investigate Riemann waves and their superpositions (i.e. Riemann  $k$ -waves), which are admitted by these systems. The main feature of the method is the introduction of new independent variables (called Riemann invariants) which remain constant on certain hyperplanes perpendicular to wave vectors associated with the initial system. This results in a reduction of the dimensionality of the problem. A number of attempts to generalize the Riemann invariants method and its various applications can be found in the recent literature of the subject (see, e.g. [12–15, 31, 38] and references therein).

At this point, we summarize the version of the GMC for constructing  $k$ -wave solutions developed progressively in [6, 18, 19, 39, 40]. Let us consider a quasilinear hyperbolic system of  $l$  first order PDEs

$$\mathcal{A}_\alpha^{\mu i}(u)u_i^\alpha = 0, \quad \mu = 1, \dots, l, \quad \alpha = 1, \dots, q, \quad i = 1, \dots, p, \quad (2.1)$$

in  $p$  independent variables  $x = (x^1, \dots, x^p) \in X \subset \mathbb{R}^p$  and  $q$  dependent variables  $u = (u^1, \dots, u^q) \in U \subset \mathbb{R}^q$ . The term  $u_i^\alpha$  denotes the first order partial derivative of  $u^\alpha$  with respect to  $x^i$ , i.e.  $u_i^\alpha \equiv \partial u^\alpha / \partial x^i$ . Here we adopt the summation convention over the repeated lower and upper indices, except in the cases in which one index is taken in brackets. The system is properly determined if  $l = q$ . All considerations have local character, that is, it suffices to look for solutions defined in a neighbourhood of  $x = 0$ . The main steps in constructing  $k$ -wave solutions can be presented as follows.

- (1) Find the real-valued functions  $\lambda^A = (\lambda_1^A, \dots, \lambda_p^A) \in X$  and  $\gamma_A = (\gamma_A^1, \dots, \gamma_A^q) \in U$  by solving the wave relation associated with the initial system (2.1),

$$(\mathcal{A}_\alpha^{\mu i}(u)\lambda_i^A)\gamma_{(A)}^\alpha = 0, \quad A = 1, \dots, k < p. \tag{2.2}$$

Thus we require that the condition

$$\text{rank}(\mathcal{A}_\alpha^{\mu i}(u)\lambda_i^A) < \min(l, q) \tag{2.3}$$

holds. We assume here the generic case in which the rank does not vary on some open subset  $\Omega \subset U$ . This step is completely algebraic.

- (2) Let us assume that we have found  $k$  linearly independent functions  $\lambda^A$  and  $\gamma_A$  which are  $C^1$  in  $\Omega$ . We postulate a form of solution  $u(x)$  of the initial system (2.1) such that all first order derivatives of  $u$  with respect to  $x^i$  are decomposable in the following way

$$\frac{\partial u^\alpha}{\partial x^i}(x) = \sum_{A=1}^k h^A(x)\gamma_A^\alpha(u)\lambda_i^A(u) \tag{2.4}$$

on some open domain  $B \subset X \times U$ . Here,  $h^A(x)$  are arbitrary functions of  $x$ . This step means that the original system (2.1) is subjected to the first order differential constraints (2.4). Thus we have to solve an overdetermined system composed of (2.1) and (2.4).

Condition (2.4), crucial to the GMC, determines the class of solutions, called Riemann  $k$ -waves, resulting from superposition of  $k$  simple waves.

- (3) Before proceeding further, we should verify whether the conditions on the vector functions  $\lambda^A$  and  $\gamma_A$ , which are necessary and sufficient for the existence of solutions of the system composed of (2.1) and (2.4), are satisfied. These conditions, in accordance with the Cartan theory of systems in involution [7], take the form

$$(i) [\gamma_A, \gamma_B] \in \text{span}\{\gamma_A, \gamma_B\}, \quad (ii) \mathcal{L}_{\gamma_B}\lambda^A \in \text{span}\{\lambda^A, \lambda^B\}, \quad A \neq B = 1, \dots, k, \tag{2.5}$$

where  $\mathcal{L}_{\gamma_B}$  denotes the Lie derivative along the vector field  $\gamma_B$  and the bracket  $[\gamma_A, \gamma_B]$  denotes the commutator of the vector fields  $\gamma_A, \gamma_B$ .

- (4) Given that conditions (2.5) are satisfied, we can choose, due to the homogeneity of the wave relation (2.2), a holonomic system for the vector fields  $\{\gamma_1, \dots, \gamma_k\}$ , by requiring a proper length for each vector  $\gamma_A$  such that

$$[\gamma_A, \gamma_B] = 0. \tag{2.6}$$

Conditions (2.6) determine a  $k$ -dimensional submanifold  $S \subset U$  which can be obtained by solving the system of PDEs

$$\frac{\partial u^\alpha}{\partial r^A} = \gamma_A^\alpha(u^1, \dots, u^q), \quad A = 1, \dots, k. \tag{2.7}$$

The solution of (2.7)

$$u = f(r^1, \dots, r^k) \tag{2.8}$$

gives the explicit parametrization of the submanifold  $S$  immersed in the space  $U$ .

- (5) Next we consider the functions  $f^*(\lambda^A)$ , that is, the functions  $\lambda^A(u)$  pulled back to the submanifold  $\mathcal{S} \subset U$ . The  $\lambda^A(u)$  become functions of the parameters  $(r^1, \dots, r^k)$  on  $\mathcal{S}$ . For simplicity of notation, we denote  $f^*(\lambda^A)$  by  $\lambda^A(r^1, \dots, r^k)$ .
- (6) Restricting equations (2.4) and (2.5) to the submanifold  $\mathcal{S}$  and using the linear independence of the vectors  $\{\gamma_1, \dots, \gamma_k\}$ , we obtain

$$\frac{\partial r^A}{\partial x^i} = h^A(x)\lambda_i^A(r^1, \dots, r^k), \tag{2.9}$$

$$\frac{\partial \lambda^A}{\partial r^B} = \alpha_B^A(r^1, \dots, r^k)\lambda^A + \beta_B^A(r^1, \dots, r^k)\lambda^B, \quad A \neq B = 1, \dots, k, \tag{2.10}$$

for some real-valued functions  $\alpha_B^A$  and  $\beta_B^A : \mathcal{S} \rightarrow \mathbb{R}$ . Here we do not use the summation convention. According to the Cartan theory of systems in involution, conditions (2.7), (2.9) and (2.10) ensure that the set of solutions of the initial system of PDEs (2.1) subjected to the differential constraints (2.4) depends on  $k$  arbitrary functions of one variable.

- (7) Next, we look for the most general class of solutions of the linear system of equations (2.10) for  $\lambda^A$  as functions of  $r^1, \dots, r^k$ . We can perform this analysis by using, for example, the Monge-Darboux method [19].
- (8) From the general solution of (2.10) for the functions  $\lambda^A$ , the solution of the system (2.9) can be derived in the implicit form

$$\lambda_i^A(r^1, \dots, r^k)x^i = \psi^A(r^1, \dots, r^k), \quad A = 1, \dots, k, \tag{2.11}$$

where  $\psi^A : \mathbb{R}^k \rightarrow \mathbb{R}$  are some functionally independent differentiable functions of  $k$  variables  $r^1, \dots, r^k$  such that

$$\frac{\partial \psi^A}{\partial r^B} = \alpha_B^A(r^1, \dots, r^k)\psi^{(A)} + \beta_B^A(r^1, \dots, r^k)\psi^{(B)}, \quad A \neq B. \tag{2.12}$$

Note that the solutions  $r^A(x)$  of (2.11) are constant on  $(p - k)$ -dimensional hyperplanes perpendicular to the wave vectors  $\lambda^A$ .

- (9) Finally, the  $k$ -wave solution of (2.1) is obtained from the explicit parametrization (2.8) of the submanifold  $\mathcal{S} \subset U$  in terms of the parameters  $r^1, \dots, r^k$ , which are now implicitly defined as functions of  $x^1, \dots, x^p$  by the solutions of the system (2.11) in the space  $X$ .

If the set of implicitly defined relations between the variables  $u^\alpha, x^i$  and  $(r^1, \dots, r^k)$ ,

$$u^\alpha = f^\alpha(r^1, \dots, r^k), \quad \lambda_i^A(r^1, \dots, r^k)x^i = \psi^A(r^1, \dots, r^k), \tag{2.13}$$

can be solved in such a way that  $r$  and  $u^\alpha$  can be given as graphs over an open subset  $\mathcal{D} \subset X$ , then the functions  $u^\alpha = f^\alpha(r^1(x), \dots, r^k(x))$  constitute a  $k$ -wave solution of the quasilinear hyperbolic system (2.1). The scalar functions  $r^A(x)$  are called the Riemann invariants. For  $p = 2$  they coincide with the classical Riemann invariants as they have been usually introduced in the literature of the subject (see, e.g. [10, 26, 41, 42]).

Finally, let us comment on the Cauchy problem for Riemann  $k$ -waves (for a detailed discussion see, e.g. [19, 30, 40]).

Let us consider  $q$  functions  $u^1(x), \dots, u^q(x)$  which take some prescribed values  $u_0(\bar{x}) = (u_0^1(\bar{x}), \dots, u_0^q(\bar{x}))$  on the hyperplane  $\mathbb{P} \subset \mathbb{R}^n$  defined by  $t = 0$ . Here, we use the notation  $x = (t, \bar{x}) \in X \subset \mathbb{R}^{n+1}$ . It was shown [19] that for  $0 < t < T$  the initial value problem for the system (2.1) has locally exactly one solution in the form of a Riemann  $k$ -wave defined implicitly by relations (2.8), (2.11) and (2.12) if the  $C^2$  function  $u_0(\bar{x})$  satisfies the following conditions.

- (i)  $u_0(\bar{x})$  is sufficiently small that there exists a time interval  $[0, T]$  in which the gradient catastrophe for a solution  $u(x)$  of (2.1) does not occur.
- (ii)  $u_0(\bar{x})$  is decomposable according to conditions (2.4), that is

$$\frac{\partial u^\alpha}{\partial x^j}(\bar{x})|_{\mathbb{P}} = \sum_{A=1}^k \xi^A(0, \bar{x}) \gamma_A^\alpha(u_0(\bar{x})) \lambda_j^A(u_0(\bar{x}))|_{\mathbb{P}} \tag{2.14}$$

on some open domain  $\mathcal{E} \subset \mathbb{P} \times U$ .

### 3. Conditional symmetries and Riemann invariants

Until now, the only way to approach the problem of superposition of many Riemann waves in multi-dimensional space was through the GMC. This method, like all other techniques of solving PDEs, has its limitations. They have motivated the authors to search for the means of constructing larger classes of multiple wave solutions expressible in terms of Riemann invariants. The natural way to do it is to look at these solutions from the point of view of group invariance properties. The feasibility and advantages of such an approach were demonstrated for certain fluid dynamic equations in [24, 25]. We have been particularly interested in the construction of nonlinear superpositions of elementary solutions (i.e., rank-1 solutions) of (2.1), and the preliminary analysis indicated that the method of conditional symmetry is an especially useful tool for this purpose.

We use the term ‘conditional symmetry’ here as introduced by P J Olver and P Rosenau [34]. It evolved from the notion of ‘nonclassical symmetry’ which had originated from the work of G Bluman and J Cole [2] and was developed by several authors (D Levi and P Winternitz [28] and Fushchych [16] among others). For a review of this subject see, e.g. [3, 8, 35] and references therein.

The method of conditional symmetry consists in supplementing the original system of PDEs with first order differential constraints for which a symmetry criterion of the given system of PDEs is identically satisfied. Under certain circumstances this augmented system of PDEs admits a larger class of Lie symmetries than the original system of PDEs. For our purpose we adapt here the version of CSM developed in [22, 23].

We now reformulate the task of constructing rank- $k$  solutions expressible in terms of Riemann invariants in the language of the group theoretical approach. Let us consider the nondegenerate system (2.1) in its matrix form

$$\mathcal{A}^1(u)u_1 + \dots + \mathcal{A}^p(u)u_p = 0, \tag{3.1}$$

where  $\mathcal{A}^1, \dots, \mathcal{A}^p$  are  $l \times q$  real-valued matrix functions of  $u$ . If we set  $l = q$ ,  $p = n + 1$  (we denote the independent variables by  $t = x^0, x^1, \dots, x^n$ ) and  $\mathcal{A}^0$  is the identity matrix, then the system has the evolutionary form

$$u_t + \sum_{j=1}^n \mathcal{A}^j(u)u_j = 0. \tag{3.2}$$

For a fixed set of  $k$  linearly independent real-valued wave vectors

$$\lambda^A(u) = (\lambda_1^A(u), \dots, \lambda_p^A(u)), \quad A = 1, \dots, k < p,$$

with

$$\ker(\lambda_i^A \mathcal{A}^i) \neq 0, \tag{3.3}$$

we define the real-valued functions  $r^A : X \times U \rightarrow \mathbb{R}$  such that

$$r^A(x, u) = \lambda_i^A(u)x^i, \quad A = 1, \dots, k. \tag{3.4}$$

These functions are Riemann invariants associated with the wave vectors  $\lambda^A$ , as introduced in the previous section.

We postulate the form of solution of (3.1) defined implicitly by the following set of relations between the variables  $u^\alpha$ ,  $x^i$  and  $r^A$

$$u = f(r^1(x, u), \dots, r^k(x, u)), \quad r^A(x, u) = \lambda_i^A(u)x^i, \quad A = 1, \dots, k. \tag{3.5}$$

Equations (3.5) determine a unique function  $u(x)$  on a neighbourhood of  $x = 0$  for any  $f : \mathbb{R}^k \rightarrow \mathbb{R}^q$ . The Jacobi matrix of equations (3.5) can be presented as

$$\partial u = (u_i^\alpha) = \left( \mathcal{I}_q - \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial u} \right)^{-1} \frac{\partial f}{\partial r} \lambda \in \mathbb{R}^{q \times p}, \tag{3.6}$$

or equivalently as

$$\partial u = \frac{\partial f}{\partial r} \left( \mathcal{I}_k - \frac{\partial r}{\partial u} \cdot \frac{\partial r}{\partial r} \right)^{-1} \lambda \in \mathbb{R}^{q \times p}, \tag{3.7}$$

where

$$\frac{\partial f}{\partial r} = \left( \frac{\partial f^\alpha}{\partial r^A} \right) \in \mathbb{R}^{q \times k}, \quad \lambda = (\lambda_i^A) \in \mathbb{R}^{k \times p}, \tag{3.8}$$

$$\frac{\partial r}{\partial u} = \left( \frac{\partial r^A}{\partial u^\alpha} \right) = \left( \frac{\partial \lambda_i^A}{\partial u^\alpha} x^i \right) \in \mathbb{R}^{k \times q}, \quad r = (r^1, \dots, r^k) \in \mathbb{R}^k, \tag{3.9}$$

and  $\mathcal{I}_q$  and  $\mathcal{I}_k$  are the  $q \times q$  and  $k \times k$  identity matrices, respectively. Applying the implicit function theorem, we obtain the following conditions ensuring that  $r^A$  and  $u^\alpha$  are expressible as graphs over some open subset  $\mathcal{D}$  of  $\mathbb{R}^p$ ,

$$\det \left( \mathcal{I}_q - \frac{\partial f}{\partial r} \cdot \frac{\partial \lambda}{\partial u} x \right) \neq 0, \tag{3.10}$$

or

$$\det \left( \mathcal{I}_k - \frac{\partial \lambda}{\partial u} x \cdot \frac{\partial f}{\partial r} \right) \neq 0. \tag{3.11}$$

The inverse matrix in (3.6) (or in (3.7)) is well defined, since

$$\frac{\partial r}{\partial u} = 0 \quad \text{at} \quad x = 0. \tag{3.12}$$

In our further considerations we assume that the conditions (3.10) or (3.11) are fulfilled, whenever applicable.

The postulated solution (3.5) is a rank- $k$  solution, since the Jacobi matrix of  $u(x)$  has a rank equal to  $k$ . Its image is a  $k$ -dimensional submanifold  $\mathcal{S}_k$  in the first jet space  $J^1 = J^1(X \times U)$ .

For a fixed set of  $k$  linearly independent wave vectors  $\{\lambda^1, \dots, \lambda^k\}$  we define another set of  $(p - k)$  linearly independent vectors

$$\xi_a(u) = (\xi_a^1(u), \dots, \xi_a^p(u))^T, \quad a = 1, \dots, p - k, \tag{3.13}$$

satisfying the orthogonality conditions

$$\lambda_i^A \xi_a^i = 0, \quad A = 1, \dots, k. \tag{3.14}$$

Then, due to (3.6) (or (3.7)), the graph of the solution  $\Gamma = \{(x, u(x))\}$  is invariant under the family of the first order differential operators

$$X_a = \xi_a^i(u) \frac{\partial}{\partial x^i}, \quad a = 1, \dots, p - k, \tag{3.15}$$

defined on  $X \times U$ . Note that the vector fields  $X_a$  do not include vectors tangent to the direction  $u$ . So, the vectors fields  $X_a$  form an Abelian distribution on  $X \times U$ , i.e.

$$[X_a, X_b] = 0, \quad a \neq b = 1, \dots, p - k. \tag{3.16}$$

Conversely, if  $u(x)$  is a  $q$ -component function defined on a neighbourhood of  $x = 0$  such that the graph  $\Gamma = \{(x, u(x))\}$  is invariant under a set of  $(p - k)$  vector fields  $X_a$  with properties (3.14), then  $u(x)$  is a solution of equations (3.5), for some  $f$ . This is so, because the set  $\{r^1, \dots, r^k, u^1, \dots, u^q\}$  constitutes a complete set of invariants of the Abelian algebra of the vector fields (3.15). This geometrically characterizes the solutions  $u(x)$  of equations (3.5).

The group-invariant solutions of the system (3.1) consist of those functions  $u = f(x)$  which satisfy both the initial system (3.1) and a set of first order differential constraints

$$\xi_a^i u_i^\alpha = 0, \quad a = 1, \dots, p - k, \quad \alpha = 1, \dots, q, \tag{3.17}$$

ensuring that the characteristics of the vectors fields  $X_a$  are equal to zero.

Note that, in general, conditions (3.17) are weaker than the DCs (2.4) required by the GMC, since the latter are submitted to the algebraic condition (2.2). Indeed, (3.17) implies

$$u_i^\alpha = \Phi_A^\alpha \lambda_i^A, \tag{3.18}$$

where  $\Phi_A^\alpha$  are real-valued matrix functions on the first jet space  $J^1 = J^1(X \times U)$ ,

$$\Phi_A^\alpha = \left[ \left( \mathcal{I}_q - \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial u} \right)^{-1} \right]_\beta^\alpha \frac{\partial f^\beta}{\partial r^A} \in \mathbb{R}^{q \times k}, \tag{3.19}$$

or

$$\Phi_A^\alpha = \frac{\partial f^\alpha}{\partial r^B} \left[ \left( \mathcal{I}_k - \frac{\partial r}{\partial u} \cdot \frac{\partial f}{\partial r} \right)^{-1} \right]_A^B \in \mathbb{R}^{q \times k}, \tag{3.20}$$

which do not necessarily satisfy the wave relation (2.2). This fact results in easing up the restrictions on initial data at  $t = 0$ , thus we are able to consider more diverse configurations of waves involved in a superposition than in the GMC case.

We now proceed to solve the overdetermined system composed of the initial system (3.1) and the DCs (3.17)

$$\mathcal{A}_\alpha^{i\mu}(u) u_i^\alpha = 0, \quad \xi_a^i(u) u_i^\alpha = 0, \quad \mu = 1, \dots, l, \quad a = 1, \dots, p - k. \tag{3.21}$$

Substituting (3.6) (or (3.7)) into (3.1) yields

$$\text{tr} \left( \mathcal{A}^\mu \left( \mathcal{I}_q - \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial u} \right)^{-1} \frac{\partial f}{\partial r} \lambda \right) = 0, \quad \mu = 1, \dots, l, \tag{3.22}$$

or

$$\text{tr} \left( \mathcal{A}^\mu \frac{\partial f}{\partial r} \left( \mathcal{I}_k - \frac{\partial r}{\partial u} \cdot \frac{\partial f}{\partial r} \right)^{-1} \lambda \right) = 0, \quad \mu = 1, \dots, l, \tag{3.23}$$

where  $\mathcal{A}^1, \dots, \mathcal{A}^l$  are  $p \times q$  matrix functions of  $u$  (i.e.,  $\mathcal{A}^\mu = (\mathcal{A}_\alpha^{\mu i}(u)) \in \mathbb{R}^{p \times q}$ ,  $\mu = 1, \dots, l$ ). For the given system of equations (2.1), the matrices  $\mathcal{A}^\mu$  are known functions of  $u$  and equations (3.22) (or (3.23)) constitute conditions on functions  $f^\alpha(r)$  and  $\lambda^A(u)$  (or, by virtue of (3.14), on  $\xi_a(u)$ ). It is convenient from a computational point of view to split  $x^i$  into  $x^{iA}$  and  $x^{i_a}$  and to choose a basis for the wave vectors  $\lambda^A$  such that

$$\lambda^A = dx^{iA} + \lambda_{i_a}^A dx^{i_a}, \quad A = 1, \dots, k, \tag{3.24}$$



where  $(i_A, i_a)$  is a permutation of  $(1, \dots, p)$ . Hence, expression (3.9) becomes

$$\frac{\partial r^A}{\partial u^\alpha} = \frac{\partial \lambda_{i_a}^A}{\partial u^\alpha} x^{i_a}. \tag{3.25}$$

Substituting (3.25) into (3.22) (or (3.23)) yields

$$\text{tr} \left( \mathcal{A}^\mu (\mathcal{I}_q - Q_a x^{i_a})^{-1} \frac{\partial f}{\partial r} \lambda \right) = 0, \quad \mu = 1, \dots, l, \tag{3.26}$$

or

$$\text{tr} \left( \mathcal{A}^\mu \frac{\partial f}{\partial r} (\mathcal{I}_k - K_a x^{i_a})^{-1} \lambda \right) = 0, \quad \mu = 1, \dots, l, \tag{3.27}$$

where

$$Q_a = \frac{\partial f}{\partial r} \eta_a \in \mathbb{R}^{q \times q}, \quad K_a = \eta_a \frac{\partial f}{\partial r} \in \mathbb{R}^{k \times k}, \quad \eta_a = \left( \frac{\partial \lambda_{i_a}^A}{\partial u^\alpha} \right) \in \mathbb{R}^{k \times q}, \tag{3.28}$$

for  $i_A$  fixed and  $i_a = 1, \dots, p - 1$ . Note that the functions  $r^A$  and  $x^{i_a}$  are functionally independent in a neighbourhood of  $x = 0$ . The matrix functions  $\mathcal{A}^\mu$ ,  $\partial f / \partial r$ ,  $Q_a$  and  $K_a$  depend on  $r$  only. Hence, equations (3.26) (or (3.27)) have to be satisfied for any value of coordinates  $x^{i_a}$ . This requirement leads to some constraints on these matrix functions.

According to the Cayley–Hamilton theorem, for any  $n \times n$  invertible matrix  $M$ , the expression  $(M^{-1} \det M)$  is a polynomial in  $M$  of order  $(n - 1)$ . Thus, using the tracelessness of the expression  $\mathcal{A}^\mu (\mathcal{I}_q - Q_a x^{i_a})^{-1} (\partial f / \partial r) \lambda$ , we can replace equations (3.26) by the following

$$\text{tr} \left( \mathcal{A}^\mu Q \frac{\partial f}{\partial r} \lambda \right) = 0, \quad \text{where } Q = \text{adj}(\mathcal{I}_q - Q_a x^{i_a}) \in \mathbb{R}^{q \times q}. \tag{3.29}$$

Here  $\text{adj} M$  denotes the classical adjoint of the matrix  $M$ . Note that  $Q$  is a polynomial of order  $(q - 1)$  in  $x^{i_a}$ . Taking (3.29) and all its partial derivatives with respect to  $x^{i_a}$  (with  $r$  fixed at  $x = 0$ ), we obtain the following conditions for the matrix functions  $f(r)$  and  $\lambda(f)$

$$\text{tr} \left( \mathcal{A}^\mu \frac{\partial f}{\partial r} \lambda \right) = 0, \quad \mu = 1, \dots, l, \tag{3.30}$$

$$\text{tr} \left( \mathcal{A}^\mu Q_{(a_1 \dots a_s)} \frac{\partial f}{\partial r} \lambda \right) = 0, \tag{3.31}$$

where  $s = 1, \dots, q - 1$  and  $(a_1, \dots, a_s)$  denotes the symmetrization over all indices in the bracket. A similar procedure can be applied to system (3.27) to yield (3.30) and

$$\text{tr} \left( \mathcal{A}^\mu \frac{\partial f}{\partial r} K_{(a_1 \dots a_s)} \lambda \right) = 0, \quad K = \text{adj}(\mathcal{I}_k - K_a x^{i_a}) \in \mathbb{R}^{k \times k}, \tag{3.32}$$

where now  $s = 1, \dots, k - 1$ . Equations (3.30) represent the initial value conditions on a surface in the space of independent variables  $X$ , given at  $x^{i_a} = 0$ . Note that equations (3.31) (or (3.32)) form the conditions required for preservation of the property (3.30) by flows represented by the vector fields (3.15). Note also that, by virtue of (3.24),  $X_a$  can be expressed in the form

$$X_a = \partial_{i_a} - \lambda_{i_a}^A \partial_{i_A}. \tag{3.33}$$

Substituting expressions (3.28) into (3.31) or (3.32) and simplifying gives the unified form

$$\text{tr} \left( \mathcal{A}^\mu \frac{\partial f}{\partial r} \eta_{(a_1} \frac{\partial f}{\partial r} \dots \eta_{a_s)} \frac{\partial f}{\partial r} \lambda \right) = 0, \quad \eta_{a_t} = \left( \frac{\partial \lambda_{a_t}^A}{\partial u^\alpha} \right) \in \mathbb{R}^{k \times q}, \quad t = 1, \dots, s, \tag{3.34}$$

where we can choose either  $\max(s) = q - 1$  or  $\max(s) = k - 1$ , whichever is more convenient.

Let us note that for  $k = 1$  the results of the two methods, CSM and GMC, overlap. This is due to the fact that conditions (2.2) and (2.7) coincide with (3.30) and conditions (2.5) and (3.34) are identically equal to zero. In this case, all rank-1 solutions correspond to single Riemann waves. However, for  $k \geq 2$  the differences between the two approaches become essential and, as we demonstrate in the following examples, the CSM can provide rank- $k$  solutions which are not Riemann  $k$ -waves as defined by the GMC.

We now introduce a change of variables on  $\mathbb{R}^p \times \mathbb{R}^q$  which allows us to rectify the vector fields  $X_a$  and simplify considerably the structure of the overdetermined system (3.21). For this system, in the new coordinates, we derive the necessary and sufficient conditions for existence of rank- $k$  solutions in the form (3.5).

Let us assume that there exists an invertible  $k \times k$  subblock

$$\Lambda = (\lambda_B^A), \quad 1 \leq A, B \leq k, \tag{3.35}$$

of the matrix  $\lambda \in \mathbb{R}^{k \times p}$ . Then the independent vector fields

$$X_{k+1} = \frac{\partial}{\partial x^{k+1}} - (\Lambda^{-1})_A^B \lambda_{k+1}^A \frac{\partial}{\partial x^B}, \dots, X_p = \frac{\partial}{\partial x^p} - (\Lambda^{-1})_A^B \lambda_p^A \frac{\partial}{\partial x^B}, \tag{3.36}$$

have the required form (3.15) for which the orthogonality conditions (3.14) are satisfied. We introduce the functions

$$\begin{aligned} \bar{x}^1 &= r^1(x, u), \dots, \bar{x}^k = r^k(x, u), \\ \bar{x}^{k+1} &= x^{k+1}, \dots, \bar{x}^p = x^p, \quad \bar{u}^1 = u^1, \dots, \bar{u}^q = u^q, \end{aligned} \tag{3.37}$$

as new coordinates on  $\mathbb{R}^p \times \mathbb{R}^q$  space which allow us to rectify the vector fields (3.36). So, we get

$$X_{k+1} = \frac{\partial}{\partial \bar{x}^{k+1}}, \dots, X_p = \frac{\partial}{\partial \bar{x}^p}. \tag{3.38}$$

The  $p$ -dimensional submanifold invariant under  $X_{k+1}, \dots, X_p$  is defined by equations of the form

$$\bar{u} = f(\bar{x}^1, \dots, \bar{x}^k), \tag{3.39}$$

for an arbitrary function  $f : \mathbb{R}^k \rightarrow \mathbb{R}^q$ . The expression (3.39) is the general solution of the invariance conditions

$$\bar{u}_{\bar{x}^{k+1}}, \dots, \bar{u}_{\bar{x}^p} = 0. \tag{3.40}$$

The initial system (3.1) described in the new coordinates  $(\bar{x}, \bar{u}) \in \mathbb{R}^p \times \mathbb{R}^q$  is, in general, a nonlinear system of first order PDEs,

$$\bar{\mathcal{A}}^i(\bar{x}, \bar{u}, \bar{u}_{\bar{x}}) \bar{u}_i = 0, \quad \text{where } \bar{\mathcal{A}}^i = \frac{\partial \bar{x}^i}{\partial x^j} \mathcal{A}^j, \quad i, j = 1, \dots, p. \tag{3.41}$$

That is, we have

$$\bar{\mathcal{A}}^1 = \frac{\partial r^1}{\partial x^i} \mathcal{A}^i, \dots, \bar{\mathcal{A}}^k = \frac{\partial r^k}{\partial x^i} \mathcal{A}^i, \quad \bar{\mathcal{A}}^{k+1} = \mathcal{A}^{k+1}, \dots, \bar{\mathcal{A}}^p = \mathcal{A}^p. \tag{3.42}$$

The Jacobi matrix in the coordinates  $(\bar{x}, \bar{u})$  takes the form

$$\frac{\partial r^i}{\partial x^j} = (\phi^{-1})_s^i \lambda_j^s \in \mathbb{R}^{k \times p}, \quad (\phi_j^i) = \left( \delta_j^i - \frac{\partial r^i}{\partial u^l} \frac{\partial \bar{u}^l}{\partial \bar{x}^j} \right) \in \mathbb{R}^{k \times k}, \tag{3.43}$$

whenever the invariance conditions (3.40) are satisfied. Augmenting the system (3.41) with the invariance conditions (3.40) leads to a quasilinear reduced system of PDEs

$$\Delta : \begin{cases} \operatorname{tr} \left( \bar{\mathcal{A}}^\mu \left( \mathcal{I}_q - \frac{\partial \bar{u}}{\partial \bar{x}} \cdot \frac{\partial r}{\partial u} \right)^{-1} \frac{\partial \bar{u}}{\partial \bar{x}} \lambda \right) = 0, & \mu = 1, \dots, l, \\ \bar{u}_{\bar{x}^{k+1}}, \dots, \bar{u}_{\bar{x}^p} = 0, \end{cases} \quad (3.44)$$

or

$$\Delta : \begin{cases} \operatorname{tr} \left( \bar{\mathcal{A}}^\mu \frac{\partial \bar{u}}{\partial \bar{x}} \left( \mathcal{I}_k - \frac{\partial r}{\partial u} \cdot \frac{\partial \bar{u}}{\partial \bar{x}} \right)^{-1} \lambda \right) = 0, & \mu = 1, \dots, l, \\ \bar{u}_{\bar{x}^{k+1}}, \dots, \bar{u}_{\bar{x}^p} = 0. \end{cases} \quad (3.45)$$

Now we present some examples which illustrate the preceding construction. If

$$\phi = \mathcal{I}_q - \frac{\partial \bar{u}}{\partial \bar{x}} \cdot \frac{\partial r}{\partial u} \in \mathbb{R}^{q \times q} \quad (3.46)$$

is a scalar matrix, then system (3.44) is equivalent to the following quasilinear system

$$B^i(\bar{u})\bar{u}_i = 0 \quad (3.47)$$

in  $k$  independent variables  $\bar{x}^1, \dots, \bar{x}^k$  and  $q$  dependent variables  $\bar{u}^1, \dots, \bar{u}^q$ , where  $B^i = \lambda_j^i \mathcal{A}^j$ . In the simplest case, when  $k = 1$ , equations (3.47) coincide with the system (3.44), i.e.

$$\lambda_i(\bar{u})\mathcal{A}^i(\bar{u})\bar{u}_1 = 0, \quad \bar{u}_2, \dots, \bar{u}_p = 0, \quad (3.48)$$

with the general solution

$$\bar{u}(\bar{x}) = f(\bar{x}^1), \quad (3.49)$$

where  $f : \mathbb{R} \rightarrow \mathbb{R}^q$  satisfies the first order ordinary differential equation

$$\lambda_i(f)\mathcal{A}^i(f)f' = 0, \quad (3.50)$$

and we have used the following notation  $f' = df/d\bar{x}^1$ .

If  $k \geq 2$  then  $\phi$  is a scalar matrix if and only if the Riemann invariants do not depend on the function  $u$ ,

$$\frac{\partial r^1}{\partial u}, \dots, \frac{\partial r^k}{\partial u} = 0. \quad (3.51)$$

Consequently, equations (3.25) and (3.51) imply that the wave vectors  $\lambda^1, \dots, \lambda^k$  are constant. Hence, this solution represents a travelling  $k$ -wave.

Consider now a more general situation when the matrix  $\phi$  does not depend on variables  $\bar{x}^{k+1}, \dots, \bar{x}^p$ , that is

$$\frac{\partial \phi}{\partial \bar{x}^{k+1}}, \dots, \frac{\partial \phi}{\partial \bar{x}^p} = 0. \quad (3.52)$$

System (3.44) is independent of  $\bar{x}^{k+1}, \dots, \bar{x}^p$  if and only if

$$\frac{\partial^2 r}{\partial u \partial \bar{x}^{k+1}}, \dots, \frac{\partial^2 r}{\partial u \partial \bar{x}^p} = 0, \quad (3.53)$$

or equivalently, due to (3.35), if and only if

$$\frac{\partial \lambda_i^A}{\partial u} = \frac{\partial \Lambda_m^A}{\partial u} (\Lambda^{-1})_n^m \lambda_i^n, \quad 1 \leq A \leq k < i < p, \quad m, n = 1, \dots, k. \quad (3.54)$$

So, it follows that

$$\frac{\partial}{\partial u} ((\Lambda^{-1})_m^A \lambda_i^m) = 0, \quad 1 \leq A \leq k < i < p. \quad (3.55)$$

Thus, equations (3.44) are independent of  $\bar{x}^{k+1}, \dots, \bar{x}^p$  if there exists a  $k \times (p - k)$  constant matrix  $C$  such that

$$(\lambda_i^A) = \Lambda \cdot C, \quad 1 \leq A \leq k < i < p. \tag{3.56}$$

In this case, (3.44) is a system (not necessarily a quasilinear one) in  $k$  independent variables  $\bar{x}^1, \dots, \bar{x}^k$  and  $q$  dependent variables  $\bar{u}^1, \dots, \bar{u}^q$ .

Let us now proceed to define some basic notions of the conditional symmetry method in the context of Riemann invariants. We associate the original system (3.1) and the invariance conditions (3.17) with the subvarieties of the solution spaces

$$\mathbb{S}_\Delta = \{(x, u^{(1)}) : \mathcal{A}_\alpha^{i\mu} u_i^\alpha = 0, \mu = 1, \dots, l\}$$

and

$$\mathbb{S}_Q = \{(x, u^{(1)}) : \xi_a^i(u) u_i^\alpha = 0, \alpha = 1, \dots, q, a = 1, \dots, p - k\},$$

respectively.

A vector field  $X_a$  is called a conditional symmetry of the original system (3.1) if  $X_a$  is tangent to  $\mathbb{S} = \mathbb{S}_\Delta \cap \mathbb{S}_Q$ , i.e.

$$\text{pr}^{(1)} X_a|_{\mathbb{S}} \in T_{(x, u^{(1)})} \mathbb{S}, \tag{3.57}$$

where  $\text{pr}^{(1)} X_a$  is the first prolongation of  $X_a$  defined on  $J^1(X \times U)$  and is given by

$$\text{pr}^{(1)} X_a = X_a - \xi_{a, u^\beta}^i u_j^\beta u_i^\alpha \frac{\partial}{\partial u_j^\alpha}, \quad a = 1, \dots, p - k, \tag{3.58}$$

and  $T_{(x, u^{(1)})} \mathbb{S}$  is the tangent space to  $\mathbb{S}$  at some point  $(x, u^{(1)}) \in J^1(X \times U)$ .

An Abelian Lie algebra  $L$  spanned by the vector fields  $X_1, \dots, X_{p-k}$  is called a conditional symmetry algebra of the original system (3.1) if the following condition

$$\text{pr}^{(1)} X_a(\mathcal{A}^i u_i)|_{\mathbb{S}} = 0, \quad a = 1, \dots, p - k, \tag{3.59}$$

is satisfied.

Note that every solution of the overdetermined system (3.21) can be represented by its graph  $\{(x, u(x))\}$ , which is a section of  $J^0$ . The conditional symmetry algebra  $L$  of (3.1) defines locally the action of the corresponding Lie group  $G$  on  $J^0$ . The symmetry group  $G$  transforms certain solutions of (3.21) into other solutions of (3.21). If the graph of a solution is preserved by  $G$  then this solution is called  $G$ -invariant.

Assume that  $L$ , spanned by the vector fields  $X_1, \dots, X_{p-k}$ , is a conditional symmetry algebra of the system (3.1). A solution  $u = f(x)$  is said to be a conditionally invariant solution of the system (3.1) if the graph  $\{(x, f(x))\}$  is invariant under the vector fields  $X_1, \dots, X_{p-k}$ .

**Proposition.** *A nondegenerate quasilinear hyperbolic system of first order PDEs (3.1) in  $p$  independent variables and  $q$  dependent variables admits a  $(p - k)$ -dimensional conditional symmetry algebra  $L$  if and only if  $(p - k)$  linearly independent vector fields  $X_1, \dots, X_{p-k}$  satisfy the conditions (3.30) and (3.34) on some neighbourhood of  $(x_0, u_0)$  of  $\mathbb{S}$ . The solution of (3.1) which are invariant under the Lie algebra  $L$  are precisely rank- $k$  solutions of the form (3.5).*

**Proof.** Let us describe the vector fields  $X_a$  in the new coordinates  $(\bar{x}, \bar{u})$  on  $\mathbb{R}^p \times \mathbb{R}^q$ . From (3.38) and (3.59) it follows that

$$\text{pr}^{(1)} X_a = X_a, \quad a = k + 1, \dots, p. \tag{3.60}$$

Hence, the symmetry criterion for  $G$  to be the symmetry group of the overdetermined system (3.44) (or (3.45)) requires that the vector fields  $X_a$  of  $G$  satisfy

$$X_a(\Delta) = 0, \tag{3.61}$$

whenever equations (3.44) (or (3.45)) hold. Thus the symmetry criterion applied to the invariance conditions (3.40) is identically equal to zero. After applying this criterion to the system (3.41) in new coordinates, carrying out the differentiation and next taking into account the conditions (3.30) and (3.34) we obtain the equations which are identically satisfied.

The converse is also true. The assumption that the system (3.1) be nondegenerate means that it is locally solvable and is of maximal rank at every point  $(x_0, u_0^{(1)}) \in \mathbb{S}$ . Therefore [36], the infinitesimal symmetry criterion is a necessary and sufficient condition for the existence of the symmetry group  $G$  of the overdetermined system (3.21). Since the vector fields  $X_a$  form an Abelian distribution on  $X \times U$ , it follows that, as we have already shown in this section, conditions (3.30) and (3.34) hold. That ends the proof, since the solutions of the overdetermined system (3.21) are invariant under the algebra  $L$  generated by  $(p - k)$  vector fields  $X_1, \dots, X_{p-k}$ . The invariants of the group  $G$  of such vector fields are provided by the functions  $\{r^1, \dots, r^k, u^1, \dots, u^q\}$ . So the general rank- $k$  solution of (3.1) takes the form (3.5).  $\square$

The expressions in equations (3.30) and (3.34) lend themselves to further simplification. Let us recall here that any  $n \times n$  matrix  $b$  is a root of the Cayley–Hamilton polynomial

$$\det(\lambda \mathcal{I}_n - b) = \lambda^n - \sum_{i=1}^n p_i(b) \lambda^{n-i}. \tag{3.62}$$

Faddeev’s approach ([17], p 87) provides a recursive method to compute the coefficients  $p_i(b)$ , based on Newton’s formulae

$$k p_k = s_k - p_1 s_{k-1} - \dots - p_{k-1} s_1, \quad s_k = \text{tr}(b^k) = \sum_{i=1}^n \lambda_i^k, \quad k = 1, \dots, n. \tag{3.63}$$

For example, one readily computes

$$\begin{aligned} p_1 &= \text{tr}(b), & p_2 &= \frac{1}{2}[\text{tr}(b^2) - (\text{tr}(b))^2], & p_3 &= \frac{1}{6}[(\text{tr}(b))^3 - 3 \text{tr}(b) \text{tr}(b^2) + 2 \text{tr}(b^3)], \\ p_4 &= \frac{1}{24}[6 \text{tr}(b^4) - (\text{tr}(b))^4 - 8 \text{tr}(b) \text{tr}(b^3) - 3(\text{tr}(b^2))^2 + 6(\text{tr}(b))^2 \text{tr}(b^2)], \\ \dots p_n &= (-1)^{n+1} \det(b). \end{aligned} \tag{3.64}$$

According to the Cayley–Hamilton theorem one has

$$b^n - \sum_{i=1}^n p_i(b) b^{n-i} = 0. \tag{3.65}$$

Using the identity (3.65), we can simplify the expressions (3.30) and (3.34) to some degree, depending on the dimension of the matrix  $b$ .

As an illustration we present the simplest case of a  $2 \times 2$  matrix, which corresponds to rank-2 solutions for  $q = 2$  unknown functions. In this case, the expressions (3.30) and (3.34) become

$$\text{tr} \left( \mathcal{A}^\mu \frac{\partial f}{\partial r} \lambda \right) = 0, \quad \mu = 1, \dots, l, \tag{3.66}$$

$$\text{tr} \left( \mathcal{A}^\mu \frac{\partial f}{\partial r} \eta_a \frac{\partial f}{\partial r} \lambda \right) = 0, \quad a = 1, \dots, p - 1. \tag{3.67}$$

Combining (3.66) and (3.67) leads to the factorized form

$$\text{tr} \left( \mathcal{A}^\mu \frac{\partial f}{\partial r} \eta_a \frac{\partial f}{\partial r} \lambda \right) = \text{tr} \left( \mathcal{A}^\mu \frac{\partial f}{\partial r} \left( \eta_a \frac{\partial f}{\partial r} - \mathcal{I}_2 \text{tr} \left( \eta_a \frac{\partial f}{\partial r} \right) \right) \lambda \right). \tag{3.68}$$

Note that for any invertible  $2 \times 2$  matrices  $M$  and  $N$ , the Cayley–Hamilton trace identity has the form

$$MN - \mathcal{I}_2 \operatorname{tr}(MN) = -(N - \mathcal{I}_2 \operatorname{tr}(N))(M - \mathcal{I}_2 \operatorname{tr}(M)). \tag{3.69}$$

Using the above equation, we rewrite (3.68) in the equivalent form

$$\begin{aligned} & \operatorname{tr} \left( \mathcal{A}^\mu \frac{\partial f}{\partial r} \left( \eta_a \frac{\partial f}{\partial r} - \mathcal{I}_2 \operatorname{tr} \left( \eta_a \frac{\partial f}{\partial r} \right) \right) \lambda \right) \\ &= -\operatorname{tr} \left( \mathcal{A}^\mu \frac{\partial f}{\partial r} \left( \frac{\partial f}{\partial r} - \mathcal{I}_2 \operatorname{tr} \left( \frac{\partial f}{\partial r} \right) \right) (\eta_a - \mathcal{I}_2 \operatorname{tr}(\eta_a)) \lambda \right), \end{aligned} \tag{3.70}$$

where the matrices  $M$  and  $N$  are identified with  $\eta_a$  and  $\partial f/\partial r$ , respectively. Since we have

$$N^2 - \operatorname{tr}(N)N = -\mathcal{I}_2 \det(N),$$

then, if  $\det(\partial f/\partial r) \neq 0$ , it follows that

$$\operatorname{tr}(\mathcal{A}^\mu (\eta_a - \mathcal{I}_2 \operatorname{tr}(\eta_a)) \lambda) = 0, \quad \eta_a = \left( \frac{\partial \lambda_i^A}{\partial u^\alpha} \right) \in \mathbb{R}^{2 \times 2}, \quad A = 1, 2. \tag{3.71}$$

For a given system (3.1) (i.e. given functions  $\mathcal{A}^\mu$ ), equations (3.71) form a bilinear system of  $l(p - 1)$  PDEs for  $2(p - 1)$  functions  $\lambda_i^A(u)$ . Thus we have eliminated the matrix term  $\partial f/\partial r$  in equations (3.67). This fact greatly facilitates our task. The proposed procedure for constructing rank-2 conditionally invariant solutions of the system (3.1) consists of the following steps.

- (1) We first look for two linearly independent real-valued wave vectors  $\lambda^1$  and  $\lambda^2$  by solving the dispersion relation (3.3) associated with the initial system (3.1).
- (2) If such wave vectors do exist, we substitute them into PDEs (3.71) and solve this system for  $\lambda^A$  in terms of  $u$ .
- (3) Next, we substitute the most general solutions for  $\lambda^1$  and  $\lambda^2$  into equations (3.66) and look for a solution  $u = f(r^1, r^2)$  of this system. Thus we obtain the explicit parametrization of the two-dimensional submanifold  $\mathcal{S} \subset U$  in terms of  $r^1$  and  $r^2$ .
- (4) We suppose that  $u = f(r^1, r^2)$  is the unique solution of PDEs (3.66). Then we restrict the wave vectors  $\lambda^A$  to the submanifold  $\mathcal{S} \subset U$ . That is, the functions  $\lambda^A(u)$  are pulled back to  $\mathcal{S}$  and become functions of the parameters  $r^1$  and  $r^2$  on  $\mathcal{S}$ . We denote the function  $f^*(\lambda^A)$  by  $\lambda^A(r^1, r^2)$ .
- (5) In this parametrization we can implicitly determine the value of the Riemann invariants for each solution  $\lambda^A$  of (3.71)

$$r^A(x, f(r^1, r^2)) = \lambda_i^A(r^1, r^2) x^i, \quad A = 1, 2. \tag{3.72}$$

- (6) Finally, we suppose that the set of implicitly defined relations (3.5) and (3.72) between  $u^\alpha, r^A$  and  $x^i$  can be solved so that the functions  $r^A$  and  $u^\alpha$  can be given as graphs over an open subset  $\mathcal{D} \subset X$ . Then the function

$$u = f(r^1(x), r^2(x)) \tag{3.73}$$

is an explicit rank-2 solution of the quasilinear hyperbolic system (3.1). The graph of this solution is invariant under  $(p - 2)$  linearly independent vector fields  $X_a$ .

As another illustration, let us consider the rank-3 case when  $q = k = 3$ . Then the condition (3.34) takes the following form

$$\operatorname{tr} \left( \mathcal{A}^\mu \frac{\partial f}{\partial r} \left[ \eta_{a_1} \frac{\partial f}{\partial r} \eta_{a_2} + \eta_{a_2} \frac{\partial f}{\partial r} \eta_{a_1} \right] \frac{\partial f}{\partial r} \lambda \right) = 0, \quad \mu = 1, \dots, l, \tag{3.74}$$

where

$$\eta_{a_j} = \left( \frac{\partial \lambda_{i a_j}^A}{\partial u^\alpha} \right) \in \mathbb{R}^{3 \times 3}, \quad j = 1, 2, \quad A = 1, 2, 3, \quad (3.75)$$

and the expressions (3.8) become

$$\frac{\partial f}{\partial r} = \left( \frac{\partial f^\alpha}{\partial r^A} \right) \in \mathbb{R}^{3 \times 3}, \quad \lambda = (\lambda_i^A(u)) \in \mathbb{R}^{3 \times p}, \quad \mathcal{A}^\mu = (\mathcal{A}_\alpha^{\mu i}) \in \mathbb{R}^{p \times 3}.$$

We introduce the notation

$$P := \eta_{a_1} \frac{\partial f}{\partial r}, \quad Q := \eta_{a_2} \frac{\partial f}{\partial r}. \quad (3.76)$$

Then, combining equations (3.30) and (3.74), we obtain

$$\text{tr} \left( \mathcal{A}^\mu \frac{\partial f}{\partial r} [PQ - \mathcal{I}_3 \text{tr}(PQ) + QP - \mathcal{I}_3 \text{tr}(QP)] \lambda \right) = 0. \quad (3.77)$$

If  $\det(\partial f / \partial r) \neq 0$  (otherwise the case  $q = 3$  can be reduced to  $q \leq 2$ ) then, similarly to the case  $q = 2$ , we can eliminate the term  $\partial f / \partial r$  from (3.30) and (3.77). The resulting expressions are still quite complicated. Nevertheless, as we show in the examples to follow, our procedure makes the construction of rank-3 solutions feasible.

#### 4. The fluid dynamics equations

At this point, we would like to illustrate the proposed approach to constructing rank- $k$  solutions with the example of the fluid dynamics equations. The fluid under consideration is assumed to be ideal, nonstationary, isentropic and compressible. We restrict our analysis to the case in which the dissipative effects, like viscosity and thermal conductivity, are negligible, and no external forces are considered. Under the above assumptions, the classical fluid dynamics model is governed by the system of equations in (3+1) dimensions of the form

$$D\rho + \rho \text{div } \vec{u} = 0, \quad D\vec{u} + \rho^{-1} \nabla p = 0, \quad DS = 0. \quad (4.1)$$

Here we have used the following notation :  $\rho$ ,  $p$  and  $S$  are the density, pressure and entropy of the fluid, respectively,  $\vec{u} = (u^1, u^2, u^3)$  is the vector field of the fluid velocity and  $D$  is the convective derivative

$$D = \frac{\partial}{\partial t} + (\vec{u} \cdot \nabla). \quad (4.2)$$

Equations (4.1) form a quasilinear hyperbolic homogeneous system of five equations in five unknown functions  $(\rho, p, \vec{u}) \in \mathbb{R}^5$ . The independent variables are denoted by  $(x^\mu) = (t, x, y, z) \in X \subset \mathbb{R}^4$ ,  $\mu = 0, 1, 2, 3$ . According to [33, 37] this system can be reduced to a hyperbolic system of four equations in four unknowns  $u = (u^\mu) = (a, \vec{u}) \in U \subset \mathbb{R}^4$  describing an isentropic ideal flow, when the sound velocity  $a$  is assumed to be a function of the density  $\rho$  only. In this case the state equation of the media  $p = f(\rho, S)$  is subjected to the differential constraints

$$\nabla p = a^2(\rho) \nabla \rho, \quad d \ln(a\rho^{-1/\kappa}) = 0, \quad (4.3)$$

where  $a^2(\rho) = \partial f / \partial \rho$ ,  $\kappa = 2(\gamma - 1)^{-1}$  and  $\gamma$  is the adiabatic exponent of the fluid. Under the assumptions (4.3), the fluid dynamics model (4.1) becomes

$$Da + \kappa^{-1} a \text{div } \vec{u} = 0, \quad D\vec{u} + \kappa a \nabla a = 0. \quad (4.4)$$

The system of equations (4.4) can be written in the equivalent matrix evolutionary form (3.2). Here  $n = 3$  and the  $4 \times 4$  matrix functions  $\mathcal{A}^1, \mathcal{A}^2$  and  $\mathcal{A}^3$  take the form

$$\mathcal{A}^i = \begin{pmatrix} u^i & \delta_{i1}\kappa^{-1}a & \delta_{i2}\kappa^{-1}a & \delta_{i3}\kappa^{-1}a \\ \delta_{i1}\kappa a & u^i & 0 & 0 \\ \delta_{i2}\kappa a & 0 & u^i & 0 \\ \delta_{i3}\kappa a & 0 & 0 & u^i \end{pmatrix}, \quad i = 1, 2, 3, \quad (4.5)$$

where  $\delta_{ij} = 1$  if  $i = j$  and 0 otherwise. The largest Lie point symmetry algebra of these equations has been already investigated in [20]. It constitutes a Galilean similitude algebra generated by 12 differential operators

$$\begin{aligned} P_\mu &= \partial_{x^\mu}, & J_k &= \epsilon_{kij}(x^i \partial_{x^j} + u^i \partial_{u^j}), & K_i &= t \partial_{x^i} + \partial_{u^i}, \quad i = 1, 2, 3, \\ F &= t \partial_t + x^i \partial_{x^i}, & G &= -t \partial_t + a \partial_a + u^i \partial_{u^i}. \end{aligned} \quad (4.6)$$

In the particular case when the adiabatic exponent is  $\gamma = 5/3$ , this algebra is generated by 13 infinitesimal differential operators, namely the 12 operators (4.6) and a projective transformation

$$C = t(t \partial_t + x^i \partial_{x^i} - a \partial_a) + (x^i - x^0 u^i) \partial_{u^i}. \quad (4.7)$$

Note that the algebras generated by (4.6) and by (4.6) with (4.7) are fibre preserving. The classification of the subalgebras of these algebras into conjugacy classes is presented in [20]. Large classes of solutions of the system (4.4), invariant and partially invariant (with the defect structure  $\delta = 1$ ), have been obtained in [21].

The wave vector  $\lambda$  can be written in the form  $(\lambda_0, \vec{\lambda})$ , where  $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$  denotes a direction of wave propagation and the eigenvalue  $\lambda_0$  is a phase velocity of a considered wave. The dispersion relation for the isentropic equations (4.4) takes the form

$$\det(\lambda_0(u)\mathcal{T}_4 + \lambda_i(u)\mathcal{A}^i(u)) = (\lambda_0 + \vec{u} \cdot \vec{\lambda})^2 [(\lambda_0 + \vec{u} \cdot \vec{\lambda})^2 - a^2 \vec{\lambda}^2] = 0. \quad (4.8)$$

Solving the dispersion equation (4.8), we obtain two types of wave vectors, namely the potential and rotational wave vectors

$$\begin{aligned} \text{(i)} \quad \lambda^E &= (\epsilon a + \vec{u} \cdot \vec{e}, -\vec{e}), & \epsilon &= \pm 1, \\ \text{(ii)} \quad \lambda^S &= ([\vec{u}, \vec{e}, \vec{m}], -\vec{e} \times \vec{m}), & |\vec{e}|^2 &= 1, \end{aligned} \quad (4.9)$$

where  $\vec{e}$  and  $\vec{m}$  are unit and arbitrary vectors, respectively. Here, equation (4.9ii) has a multiplicity of 2. The quantity  $[\vec{u}, \vec{e}, \vec{m}]$  denotes the determinant of the matrix based on these vectors, i.e.  $[\vec{u}, \vec{e}, \vec{m}] = \det(\vec{u}, \vec{e}, \vec{m})$ . Several classes of  $k$ -wave solutions of the isentropic system (4.4), obtained via the GMC, are known [5, 39]. Applying the CSM to this system allows us to compare the effectiveness of the two approaches.

### 5. Rank-1 solutions of the fluid dynamics equations

Analysing the rank-1 solutions associated with the wave vectors  $\lambda^E$  and  $\lambda^S$  we consider separately two cases.

In the first case, the potential wave vectors are the nonzero multiples of

$$\lambda^E = (\epsilon a + \vec{e} \cdot \vec{u}, -\vec{e}), \quad |\vec{e}|^2 = 1, \quad \epsilon = \pm 1.$$

The corresponding vector fields  $X_i$  and Riemann invariant  $r(x, u)$  are given by

$$X_i = -(a + \vec{e} \cdot \vec{u})^{-1} e_i \frac{\partial}{\partial t} + \frac{\partial}{\partial x^i}, \quad i = 1, 2, 3, \quad r(x, u) = (a + \vec{e} \cdot \vec{u})t - \vec{e} \cdot \vec{x}.$$



We can now consider rank-1 potential solutions, invariant under the vector fields  $\{X_1, X_2, X_3\}$ . The change of coordinates

$$\begin{aligned} \bar{t} &= t, & \bar{x}^1 &= r(x, u), & \bar{x}^2 &= x^2, & \bar{x}^3 &= x^3, \\ \bar{a} &= a, & \bar{u}^1 &= u^1, & \bar{u}^2 &= u^2, & \bar{u}^3 &= u^3, \end{aligned} \quad (5.1)$$

on  $\mathbb{R}^4 \times \mathbb{R}^4$  transforms the fluid dynamics equations (4.4) into the system

$$\frac{\partial \bar{a}}{\partial \bar{x}^1} = \kappa^{-1} e_i \frac{\partial \bar{u}^i}{\partial \bar{x}^1}, \quad \frac{\partial \bar{u}^i}{\partial \bar{x}^1} = \kappa e_i \frac{\partial \bar{a}}{\partial \bar{x}^1}, \quad i = 1, 2, 3, \quad (5.2)$$

with the invariance conditions

$$\bar{a}_{\bar{t}} = \bar{a}_{\bar{x}^j} = 0, \quad \bar{u}_{\bar{t}}^\alpha = \bar{u}_{\bar{x}^j}^\alpha = 0, \quad j = 2, 3, \quad \alpha = 1, 2, 3.$$

If the unit vector  $\vec{e}$  has the form  $\vec{e} = (\cos \bar{u}^1 \sin \bar{u}^2, \cos \bar{u}^1 \cos \bar{u}^2, \sin \bar{u}^1)$ , then the general rank-1 solution is given by

$$\begin{aligned} \bar{a}(\bar{t}, \bar{x}) &= \kappa^{-1} \bar{x}^1 + a^0, & \bar{u}^1(\bar{t}, \bar{x}) &= -\ln |C \cos \bar{u}^2|, & C &\in \mathbb{R} \\ \bar{u}^2(\bar{t}, \bar{x}) &= \bar{u}^2(\bar{x}^1), & \bar{u}^3(\bar{t}, \bar{x}) &= -\int_0^{\bar{u}^2} \tan(\ln |C \cos s|) \cos s \, ds. \end{aligned}$$

In particular, if  $\vec{e}$  is a constant unit vector, then we can integrate (5.2) and the solution is defined implicitly by equations

$$\bar{a}(\bar{t}, \bar{x}) = \bar{a}(\bar{x}^1), \quad \bar{u}^i(\bar{t}, \bar{x}) = \kappa e_i \bar{a}(\bar{x}^1) + C_i, \quad C_i \in \mathbb{R}, \quad i = 1, 2, 3.$$

If we choose  $\bar{a} = A_1 \bar{x}^1$ ,  $A_1 \in \mathbb{R}$  and  $C_i = 0$ , then the explicit invariant solution has the form

$$a(t, x) = [A_1(1 + \kappa)t - 1]^{-1} A_1 \vec{e} \cdot \vec{x}, \quad \bar{u}(t, x) = [A_1(1 + \kappa)t - 1]^{-1} \kappa A_1 (\vec{e} \cdot \vec{x}) \vec{e}. \quad (5.3)$$

Note that if the characteristics of one family associated with the eigenvalue  $\lambda_0 = a + \vec{e} \cdot \vec{u}$  intersect, then we can choose a particular value of time interval  $[t_0, T]$ , where  $T = (A_1(1 + \kappa))^{-1}$ , in order to exclude the possibility of a gradient catastrophe. Hence, if the initial data are sufficiently small at  $t = t_0$ , then the solution (5.3) remains a rank-1 solution for the time  $t \in [t_0, T]$ , and no discontinuities (e.g., shock waves) can appear.

In the second case, we fix a rotational wave vector

$$\lambda^S = ([\vec{u}, \vec{e}, \vec{m}], -\vec{e} \times \vec{m}), \quad |\vec{e}|^2 = 1,$$

and the corresponding vector fields (3.36) are given by

$$X_i = [\vec{u}, \vec{e}, \vec{m}]^{-1} (\vec{e} \times \vec{m})_i \frac{\partial}{\partial t} + \frac{\partial}{\partial x^i}, \quad i = 1, 2, 3.$$

Hence, the Riemann invariant associated with  $\lambda^S$  has the form

$$r(x, u) = [\vec{u}, \vec{e}, \vec{m}]t - [\vec{x}, \vec{e}, \vec{m}]. \quad (5.4)$$

After substituting (5.4) into (5.1), the change of coordinates transforms the initial system (4.4) into the overdetermined system composed of the following equations

$$\left[ \frac{\partial \vec{u}}{\partial \bar{x}^1}, \vec{e}, \vec{m} \right] = 0, \quad (e_i m_j - e_j m_i) \frac{\partial \bar{a}}{\partial \bar{x}^1} = 0, \quad i \neq j = 1, 2, 3, \quad (5.5)$$

and the invariance conditions

$$\bar{a}_{\bar{t}} = \bar{a}_{\bar{x}^j} = 0, \quad \bar{u}_{\bar{t}}^\alpha = \bar{u}_{\bar{x}^j}^\alpha = 0, \quad j = 2, 3, \quad \alpha = 1, 2, 3. \quad (5.6)$$

Hence, the sound velocity is constant ( $a = a_0$ ). If  $\vec{e}$  and  $\vec{m}$  are constant vectors such that  $(e_1 m_2 - e_2 m_1) \neq 0$ , then we can integrate the system composed of (5.5) and (5.6). The explicit solution is given by

$$\begin{aligned} \bar{a}(\bar{t}, \bar{x}) &= a_0, & \bar{u}^1(\bar{t}, \bar{x}) &= \bar{u}^1(\bar{x}^1), & \bar{u}^2(\bar{t}, \bar{x}) &= \bar{u}^2(\bar{x}^1), \\ \bar{u}^3(\bar{t}, \bar{x}) &= (e_1 m_2 - e_2 m_1)^{-1} [C - (e_2 m_3 - e_3 m_2) \bar{u}^1 - (e_3 m_1 - e_1 m_3) \bar{u}^2], \end{aligned} \quad (5.7)$$

where  $\bar{u}^1$  and  $\bar{u}^2$  are arbitrary functions of the Riemann invariant, which takes the form

$$r(x, u) = Ct - [\vec{x}, \vec{e}, \vec{m}]. \tag{5.8}$$

As expected, this result coincides with the solution obtained through the GMC [39]. The presence of arbitrary functions in the obtained solution allows us to find bounded solutions valid for all time  $t > 0$ . For example, the bounded bump-type solution  $\bar{u}^i = \text{sech}(A_{(i)}r^i)$ ,  $i = 1, 2$ , contains no discontinuities.

### 6. Rank-2 solutions

The construction approach outlined in section 3 has been applied to the isentropic flow equations (4.4) in order to obtain rank-2 and rank-3 solutions (the latter are presented in the next section). In the case of rank-2 solutions, in order to facilitate computations, we assume that the directions of wave propagation  $\vec{\lambda}^A$  are constant, but not their phase velocities  $\lambda_0^A$ .

After considering all possible combinations of the potential and rotational wave vectors ( $\lambda^{E_i}$  and  $\lambda^{S_i}$ , respectively,  $i = 1, 2, 3$ ) we found eight cases compatible with conditions (3.30) and (3.34), leading to eight different classes of solutions. These solutions, in their general form, possess some degree of freedom, that is, depend on one or two arbitrary functions of one or two variables (Riemann invariants), depending on the case. This arbitrariness allows us to change the geometrical properties of the governed fluid flow in such a way as to exclude the presence of singularities. This fact is of a special significance here since, as is well known [4, 9, 41], in most cases, even for arbitrary smooth and sufficiently small initial data at  $t = t_0$ , the magnitude of the first derivatives of Riemann invariants becomes unbounded in some finite time  $T$ ; thus, solutions expressible in terms of Riemann invariants usually admit a gradient catastrophe. Nevertheless, we have been able to demonstrate that it is still possible in these cases to construct bounded solutions and, in particular, soliton-like solutions, through the proper selection of the arbitrary function(s) appearing in the general solution. To this purpose we submit this arbitrary function(s), say  $v$ , to the differential constraint in the form of the nonlinear Klein–Gordon equation in two independent variables  $r^1$  and  $r^2$

$$\square(r^1, r^2)v = cv^5, \quad c \in \mathbb{R} \tag{6.1}$$

which is known to possess rich families of soliton-type solutions (see, e.g. [1, 43]). Equation (6.1) can be reduced to a second order ODE for  $v$  as a function of a Riemann invariant and can very often be explicitly integrated. The analysis of the singularity structure of these ODEs allows us to select soliton-like solutions for  $v$  which, in turn, in many cases, lead to the same type of rank-2 and rank-3 solutions of the system (4.4). Among them we have various types of algebraic soliton-like solutions (admitting no singularity other than poles), kinks, bumps and doubly periodic solutions which are expressed in terms of Jacobi elliptic functions.

Below we list the obtained rank-2 solutions. Some of the general solutions found (both rank-2 and rank-3 in the next section) coincide with the ones obtained earlier by means of the GMC. Nevertheless, we list them all since we derive from them the particular bounded solutions, which, to our knowledge, are all new.

For convenience, we denote by  $(E_i E_j, E_i S_j, S_i S_j, E_i E_j E_k, \dots, i, j, k = 1, 2, 3)$  the solutions which result from nonlinear superpositions of rank-1 solutions associated with given wave vectors  $\lambda^{E_i}$  or  $\lambda^{S_i}$ . The sign (+ or -) coincides with the value of  $\epsilon = \pm 1$  in equation (4.9).

Case  $(E_1 E_2)$ . We first discuss the superposition of two potential rank-1 solutions  $E_i$  for which the wave vectors have the form

$$\lambda^{E_i^\pm} = (\epsilon a + \vec{e}^i \cdot \vec{u}, -\vec{e}^i), \quad |\vec{e}^i|^2 = 1, \quad i = 1, 2, \quad \epsilon = \pm 1. \tag{6.2}$$

We assume that the wave vectors  $\lambda^{E_1^\pm}$  and  $\lambda^{E_2^\pm}$  are linearly independent. The corresponding vector fields (3.36) are given by

$$X_1 = \frac{\partial}{\partial x^2} - \frac{\sigma_2}{\beta_1} \frac{\partial}{\partial t} - \frac{\beta_2}{\beta_1} \frac{\partial}{\partial x^1}, \quad X_2 = \frac{\partial}{\partial x^3} - \frac{\sigma_3}{\beta_1} \frac{\partial}{\partial t} - \frac{\beta_3}{\beta_1} \frac{\partial}{\partial x^1}, \tag{6.3}$$

with

$$\begin{aligned} \beta_i &= e_i^2(a + \vec{e}^1 \cdot \vec{u}) - e_i^1(a + \vec{e}^2 \cdot \vec{u}), & i &= 1, 2, 3, \\ \sigma_j &= e_j^1 e_j^2 - e_j^2 e_j^1, & j &= 2, 3, & \epsilon &= 1. \end{aligned} \tag{6.4}$$

The nonscattering rank-2 potential solution  $(E_1^+ E_2^+)$  has the form

$$a = a_1(r^1) + a_2(r^2), \quad \vec{u} = \kappa(a_1(r^1)\vec{e}^1 + a_2(r^2)\vec{e}^2), \tag{6.5}$$

where  $a_1$  and  $a_2$  are arbitrary functions of the Riemann invariants

$$\begin{aligned} r^1(x, u) &= (1 + \kappa)a_1(r^1)t - \vec{e}^1 \cdot \vec{x}, & |\vec{e}^1|^2 &= 1, \\ r^2(x, u) &= (1 + \kappa)a_2(r^2)t - \vec{e}^2 \cdot \vec{x}, & |\vec{e}^2|^2 &= 1, \end{aligned} \tag{6.6}$$

respectively, and the wave vectors  $\vec{e}^1$  and  $\vec{e}^2$  have to satisfy the condition

$$\vec{e}^1 \cdot \vec{e}^2 + \kappa^{-1} = 0. \tag{6.7}$$

Equation (6.7) holds if and only if the angle  $\varphi$  between these vectors is

$$\cos \varphi = \frac{1}{2}(1 - \gamma). \tag{6.8}$$

This solution represents a Riemann double wave. Here, the rank-1 solutions  $E_i^+, i = 1, 2$ , do not influence each other (they superpose linearly). This result coincides with the one obtained earlier by means of the GMC [39].

- (i) In the particular case when  $a_i(r^i) = -A_i r^i, A_i \in \mathbb{R}, i = 1, 2$ , the solution (6.5) takes the explicit form

$$\begin{aligned} a &= \frac{A_1 \vec{e}^1 \cdot \vec{x}}{1 + (1 + \kappa)A_1 t} + \frac{A_2 \vec{e}^2 \cdot \vec{x}}{1 + (1 + \kappa)A_2 t} \\ \vec{u} &= \frac{\kappa A_1 \vec{e}^1 \cdot \vec{x}}{1 + (1 + \kappa)A_1 t} \vec{e}^1 + \frac{\kappa A_2 \vec{e}^2 \cdot \vec{x}}{1 + (1 + \kappa)A_2 t} \vec{e}^2, \end{aligned} \tag{6.9}$$

which admits the gradient catastrophe at the time  $t = \min(A_i^{-1}(1 + \kappa)^{-1}), i = 1, 2$ . Hence, some discontinuities can occur e.g. shock waves which correspond to the formation of a condensation jump from the compression waves related to  $\lambda^{E_i^+}$ .

- (ii) The following bounded solution can be obtained using the DC (6.1):

$$\begin{aligned} a &= \sum_{i=1}^2 A_i r^i (1 + B_i (r^i)^2)^{-1/2}, & A_i, B_i &\in \mathbb{R}, & B_i &> 0, \\ \vec{u} &= \kappa \left[ \sum_{i=1}^2 A_i r^i (1 + B_i (r^i)^2)^{-1/2} \vec{e}^i \right], \end{aligned} \tag{6.10}$$

where the Riemann invariants are given by

$$r^i = [(1 + \kappa)A_i r^i (1 + B_i (r^i)^2)^{-1/2}]t - \vec{e}^i \cdot \vec{x}, \quad i = 1, 2. \tag{6.11}$$

The result (6.10) represents an algebraic kink-type solution which is bounded for  $t > 0$  while each  $r^i$  possesses a discontinuity at time  $T = (A_i(1 + \kappa))^{-1}$ .

Case  $(E_1 S_2)$ . In the mixed case  $(E_1^+ S_2)$ , we consider the superposition of the rank-1 potential solution  $E_1^+$  with the rank-1 rotational solution  $S_2$  associated respectively with the wave vectors

$$\begin{aligned} \lambda^{E_1^+} &= (a + \vec{e}^1 \cdot \vec{u}, -\vec{e}^1), \\ \lambda^{S_2} &= ([\vec{u}, \vec{e}^2, \vec{m}^2], -\vec{e}^2 \times \vec{m}^2), \quad |\vec{e}^i|^2 = 1, \quad i = 1, 2. \end{aligned} \tag{6.12}$$

The vector fields (3.36) corresponding to the wave vectors (6.12) are

$$X_1 = \frac{\partial}{\partial x^2} - \frac{\sigma_2}{\beta_1} \frac{\partial}{\partial t} - \frac{\beta_2}{\beta_1} \frac{\partial}{\partial x^1}, \quad X_2 = \frac{\partial}{\partial x^3} - \frac{\sigma_3}{\beta_1} \frac{\partial}{\partial t} - \frac{\beta_3}{\beta_1} \frac{\partial}{\partial x^1}, \tag{6.13}$$

where

$$\begin{aligned} \beta_i &= -(\vec{e}^2 \times \vec{m}^2)_i (a + \vec{e}^1 \cdot \vec{u}) + e_i^1 [\vec{u}, \vec{e}^2, \vec{m}^2], \quad i = 1, 2, 3, \\ \sigma_j &= -e_j^1 (\vec{e}^2 \times \vec{m}^2)_j + e_j^1 (\vec{e}^2 \times \vec{m}^2)_1, \quad j = 2, 3. \end{aligned} \tag{6.14}$$

The invariant nonscattering rank-2 solution  $(E_1^+ S_2)$  has the form

$$a = a_1(r^1) + a_0, \quad \vec{u} = \kappa a_1(r^1) \vec{e}^1 + \vec{u}_2(r^2), \tag{6.15}$$

where

$$[\vec{u}_2, \vec{e}^2, \vec{m}^2] = C_2, \quad |\vec{e}^i|^2 = 1, \quad i = 1, 2, \tag{6.16}$$

and  $a_1$  and  $u_2^1$  are any differentiable functions of  $r^1$  and  $r^2$ , respectively, and the relation  $u_2^3(r^2) = C_1 u_2^1(r^2)$  holds. Here,  $a_0, C_1, C_2 \in \mathbb{R}$  and  $\vec{m}^2$  is an arbitrary constant vector. The wave vector  $\lambda^{S_2}$  takes the form

$$\lambda^{S_2} = (C_2, -(e_1^1 e_3^1 + C_1(1 - (e_1^1)^2)), -e_2^1(e_3^1 - C_1 e_1^1), (1 - (e_3^1)^2 + C_1 e_1^1 e_3^1)). \tag{6.17}$$

From (6.15), (6.16) and (6.17), we get

$$[\vec{e}^1, \vec{e}^2, \vec{m}^2] = 0, \tag{6.18}$$

so the vector  $\vec{\lambda}^{E_1^+} = -\vec{e}^1$  is orthogonal to  $\vec{\lambda}^{S_2} = -\vec{e}^2 \times \vec{m}^2$ . Hence, the Riemann invariants are given by

$$\begin{aligned} r^1 &= ((1 + \kappa)a_1(r^1) + C_2(C_1 e_1^1 - e_3^1)^{-1})t - \vec{e}^1 \cdot \vec{x}, \\ r^2 &= C_2 t - (e_1^1 e_3^1 + C_1(1 - (e_1^1)^2))x^1 - e_2^1(e_3^1 - C_1 e_1^1)x^2 + (1 - (e_3^1)^2 + C_1 e_1^1 e_3^1)x^3. \end{aligned} \tag{6.19}$$

This solution represents a Riemann double wave.

- (i) An explicit form of solution (6.15) can be found when  $\vec{e}^1 = \vec{e}^2 = (\cos \varphi, \sin \varphi, 0)$  and  $\vec{m}^2 = (\sin \varphi, -\cos \varphi, C_1 \sin \varphi)$ ,  $C_1 \in \mathbb{R}$ , and we choose  $a_1(r^1) = A_1 r^1$ ,  $A_1 \in \mathbb{R}$ . The Riemann invariants are now given by

$$\begin{aligned} r^1 &= \frac{C_2 t + x^1 C_1 \cos^2 \varphi + x^2 C_1 \cos \varphi \sin \varphi}{(C_1 \cos \varphi)(A_1(1 + \kappa)t - 1)}, \\ r^2 &= C_2 t - C_1 x^1 \sin^2 \varphi + x^2 C_1 \sin \varphi \cos \varphi + x^3, \end{aligned} \tag{6.20}$$

and the solution becomes

$$\begin{aligned} a &= A_1 \frac{C_2 t + C_1 x^1 \cos^2 \varphi + C_1 x^2 \cos \varphi \sin \varphi}{(C_1 \cos \varphi)(A_1(1 + \kappa)t - 1)}, \quad u^3 = C_1 u_2^1(r^2), \\ u^1 &= \frac{\kappa A_1 (C_2 t + C_1 x^1 \cos^2 \varphi + C_1 x^2 \sin \varphi \cos \varphi)}{C_1 (A_1(1 + \kappa)t - 1)} + u_2^1(r^2), \\ u^2 &= \frac{\kappa A_1}{C_1} (C_2 \tan \varphi t + C_1 x^1 \sin \varphi \cos \varphi + C_1 x^2 \sin^2 \varphi) - \frac{C_2}{C_1 \sin \varphi \cos \varphi} - u_2^1(r^2) \cot \varphi, \end{aligned} \tag{6.21}$$

where  $u_2^1(r^2)$  is an arbitrary function of  $r^2$ . Note that  $a$  and  $u^1$  admit the gradient catastrophe at the time  $T = (A_1(1 + \kappa))^{-1}$ .

(ii) Another interesting case of a conditionally invariant solution occurs when we impose condition (6.1) on the functions  $a_1$  and  $u_2^1$ . Then the solution is bounded and represents a solitary double wave of the type  $(E_1^+ S_2)$

$$a = A_1(1 + B_1(r^1 - 1)^2)^{-1/2} + a_0, \quad A_1, B_1, C_1 \in \mathbb{R}, \quad B_1 > 0, \tag{6.22}$$

$$\vec{u} = \kappa A_1(1 + B_1(r^1 - 1)^2)^{-1/2} \vec{e}^1 + (u_2^1(r^2), E_2 u_2^1(r^2) + F_2, C_1 u_2^1(r^2))^T,$$

where

$$u_2^1(r^2) = A_2(1 + B_2 \cosh D_2(r^2 - 1))^{-1/2}, \quad A_2, B_2, D_2 \in \mathbb{R}, \quad B_2 > 0,$$

$$E_2 = -(e_2^1(C_1 e_1^1 - e_3^1))^{-1} (C_1 e_3^1 + e_1^1)(C_1 e_1^1 - e_3^1), \quad F_2 = C_2(e_2^1(C_1 e_1^1 - e_3^1))^{-1}. \tag{6.23}$$

The Riemann invariants take the form

$$r^1 = (1 + \kappa)(A_1(1 + B_1(r^1 - 1)^2)^{-1/2} + C_2(C_1 e_1^1 - e_3^1)^{-1})t - \vec{e}^1 \cdot \vec{x},$$

$$r^2 = C_2 t - (e_1^1 e_3^1 + C_1(1 - (e_1^1)^2))x^1 - e_2^1(e_3^1 - C_1 e_1^1)x^2 + (1 - (e_3^1)^2 + C_1 e_1^1 e_3^1)x^3. \tag{6.24}$$

The solution remains bounded even though the function  $r^1$  admits the gradient catastrophe at the time  $T = (1 + B_1)^{3/2}[(1 + \kappa)A_1 B_1]^{-1}$ .

Case  $(S_1 S_2)$ . (i) Let us assume that

$$\vec{e}^1 = (0, 0, 1), \quad \vec{m}^1 = (0, 1, 0), \quad \vec{e}^2 = (1, 0, 0), \quad \vec{m}^2 = (0, 0, 1).$$

Then the wave vectors (4.9ii) are given by  $\lambda^{S_1} = (-u^1, 1, 0, 0)$  and  $\lambda^{S_2} = (-u^2, 0, 1, 0)$  and are linearly independent. So we are looking for rank-2 solution  $(S_1 S_2)$  invariant under the vector fields

$$X_1 = \frac{\partial}{\partial t} + u^1 \frac{\partial}{\partial x^1} + u^2 \frac{\partial}{\partial x^2}, \quad X_2 = \frac{\partial}{\partial x^3}. \tag{6.25}$$

The corresponding Riemann invariants are

$$r^1(x, u) = x^1 - u^1 t, \quad r^2(x, u) = x^2 - u^2 t. \tag{6.26}$$

The change of coordinates

$$\begin{aligned} \bar{t} &= t, & \bar{x}^1 &= x^1 - u^1 t, & \bar{x}^2 &= x^2 - u^2 t, & \bar{x}^3 &= x^3, \\ \bar{a} &= a, & \bar{u}^1 &= u^1, & \bar{u}^2 &= u^2, & \bar{u}^3 &= u^3, \end{aligned} \tag{6.27}$$

transforms the system (4.4) in this case into the equations

$$\begin{aligned} \frac{\partial \bar{u}^1}{\partial \bar{x}^1} + \frac{\partial \bar{u}^2}{\partial \bar{x}^2} &= 0, & \frac{\partial \bar{u}^1}{\partial \bar{x}^1} \frac{\partial \bar{u}^2}{\partial \bar{x}^2} - \frac{\partial \bar{u}^1}{\partial \bar{x}^2} \frac{\partial \bar{u}^2}{\partial \bar{x}^1} &= 0, \\ \frac{\partial \bar{a}}{\partial \bar{x}^1} = \frac{\partial \bar{a}}{\partial \bar{x}^2} = \frac{\partial \bar{a}}{\partial \bar{x}^3} &= 0, & \frac{\partial \bar{u}^i}{\partial \bar{x}^3} &= 0, \quad i = 1, 2, 3. \end{aligned} \tag{6.28}$$

The solution of system (6.28) has the form

$$\bar{a} = a_0, \quad \bar{u}^1(\bar{t}, \bar{x}) = -\frac{\partial \psi}{\partial \bar{x}^2}, \quad \bar{u}^2(\bar{t}, \bar{x}) = \frac{\partial \psi}{\partial \bar{x}^1}, \quad \bar{u}^3(\bar{t}, \bar{x}) = \bar{u}^3(\bar{x}^1, \bar{x}^2), \tag{6.29}$$

where the function  $\psi$  satisfies the homogeneous Monge-Ampère equation

$$\psi_{\bar{x}^1 \bar{x}^1} \psi_{\bar{x}^2 \bar{x}^2} - \psi_{\bar{x}^1 \bar{x}^2}^2 = 0, \tag{6.30}$$

and  $\bar{u}^3$  is an arbitrary function of two variables. Note that this solution has rank 2 but it is not a Riemann double wave.

(i) The proper selection of the function  $\psi$  transforms the solution (6.29) into

$$\begin{aligned}
 a(t, x) &= a_0, & u^1 &= (1 - n) \left( \frac{x^1 - u^1 t}{x^2 - u^2 t} \right)^n, & n &\in \mathbb{Z} \setminus \{1\}, \\
 u^2 &= -n \left( \frac{x^2 - u^2 t}{x^1 - u^1 t} \right)^{1-n}, & u^3(t, x) &= u^3(x^1 - u^1 t, x^2 - u^2 t).
 \end{aligned}
 \tag{6.31}$$

For  $n = 2$ , we obtain an explicit solution of the form

$$\begin{aligned}
 a &= a_0, & u^1 &= -2^{-1} t^{-2} [x^1 t + (x^2)^2 \pm x^2 ((x^2)^2 + 4t x^1)^{1/2}], \\
 u^2 &= t^{-1} [x^2 \pm ((x^2)^2 + 4t x^1)^{1/2}], & u^3 &= u^3(x^1 - u^1 t, x^2 - u^2 t)
 \end{aligned}
 \tag{6.32}$$

with a singularity at  $t = 0$ .

(ii) Another example worth considering is the case when fluid velocity can be decomposed as follows  $\vec{u} = \vec{u}_1(r^1) + \vec{u}_2(r^2)$ . Then we get the scattering nonsingular rank-2 solution

$$\begin{aligned}
 u^1 &= \frac{(C_1 - \lambda_2^1 u_1^2(r^1) - \lambda_3^1 u_1^3(r^1))}{\lambda_1^1} - \left( \frac{\lambda_3^2}{\lambda_1^2} \eta + \frac{\lambda_2^2}{\lambda_1^2} \right) u_2^2(r^2) + \frac{C_2}{\lambda_1^2}, & C_1, C_2 &\in \mathbb{R}, \\
 u^2 &= u_1^2(r^1) + u_2^2(r^2), & u^3 &= u_1^3(r^1) + \eta u_2^2(r^2), & a &= a_0, & \eta &= \frac{\lambda_1^2 \lambda_2^1 - \lambda_1^1 \lambda_2^2}{\lambda_1^1 \lambda_3^2 - \lambda_1^3 \lambda_1^2},
 \end{aligned}
 \tag{6.33}$$

where we introduced the notation  $\lambda^{S_i} = (\lambda_0^i, \vec{\lambda}^i)$ ,  $i = 1, 2$ . The above solution is invariant under the vector fields

$$X_1 = \frac{\partial}{\partial x^2} - \frac{\sigma_2}{\beta_1} \frac{\partial}{\partial t} - \frac{\beta_2}{\beta_1} \frac{\partial}{\partial x^1}, \quad X_2 = \frac{\partial}{\partial x^3} - \frac{\sigma_3}{\beta_1} \frac{\partial}{\partial t} - \frac{\beta_3}{\beta_1} \frac{\partial}{\partial x^1},
 \tag{6.34}$$

with

$$\sigma_i = \lambda_1^1 \lambda_i^2 - \lambda_i^1 \lambda_1^2, \quad \beta_j = \lambda_j^2 [\vec{u}, \vec{e}^1, \vec{m}^1] - \lambda_j^1 [\vec{u}, \vec{e}^2, \vec{m}^2], \quad i = 2, 3, \quad j = 1, 2, 3.
 \tag{6.35}$$

Here  $u_1^2$  and  $u_1^3$  are arbitrary functions of  $r^1$ ,  $u_2^2$  is an arbitrary function of  $r^2$  and  $C_i = [\vec{u}_i(r^i), \vec{e}^i, \vec{m}^i]$ ,  $i = 1, 2$ . The Riemann invariants take the form

$$\begin{aligned}
 r^1 &= (C_1 + C_2 \lambda_1^1 / \lambda_1^2) t - \vec{\lambda}^1 \cdot \vec{x}, \\
 r^2 &= \left( C_2 + \frac{\lambda_1^2}{\lambda_1^1} C_1 + \left( \lambda_2^2 - \frac{\lambda_1^2 \lambda_2^1}{\lambda_1^1} \right) u_1^2(r^1) + \left( \lambda_3^2 - \frac{\lambda_1^2 \lambda_3^1}{\lambda_1^1} \right) u_1^3(r^1) \right) t - \vec{\lambda}^2 \cdot \vec{x}.
 \end{aligned}
 \tag{6.36}$$

Note that the Riemann invariant  $r^2$  depends functionally on  $r^1$ . This means that the interacting waves influence each other and superpose nonlinearly. The result is a Riemann double wave.

(iii) By submitting the arbitrary functions  $u_1^2, u_1^3$  and  $u_2^2$  appearing in (6.33) to the DC (6.1), we can construct the rank-2 algebraic kink-type solution of the form

$$\begin{aligned}
 u^1 &= (\lambda_1^1)^{-1} [C_1 - \lambda_2^1 A_2 r^1 (1 + B_2 (r^1)^2)^{-1/2} - \lambda_3^1 A_3 r^1 (1 + B_3 (r^1)^2)^{-1/2}] \\
 &\quad - (\lambda_1^2)^{-1} [-C_2 + (\lambda_3^2 \eta + \lambda_2^2) A_1 r^2 (1 + B_1 (r^2)^2)^{-1/2}], & A_i, B_i &\in \mathbb{R}, \\
 u^2 &= A_1 r^2 (1 + B_1 (r^2)^2)^{-1/2} + A_2 r^1 (1 + B_2 (r^1)^2)^{-1/2}, & B_i &> 0, \quad i = 1, 2, 3, \\
 u^3 &= A_3 r^1 (1 + B_3 (r^1)^2)^{-1/2} + \eta A_1 r^2 (1 + B_1 (r^2)^2)^{-1/2}, & a &= a_0,
 \end{aligned}
 \tag{6.37}$$

where the Riemann invariants are given by

$$\begin{aligned}
 r^1 &= \left( C_1 + C_2 \frac{\lambda_1^1}{\lambda_1^2} \right) t - \vec{\lambda}^1 \cdot \vec{x}, \\
 r^2 &= \left[ C_2 + C_1 \frac{\lambda_1^2}{\lambda_1^1} + \left( \lambda_2^2 - \frac{\lambda_1^2 \lambda_2^1}{\lambda_1^1} \right) A_2 r^1 (1 + B_2 (r^1)^2)^{-1/2} \right. \\
 &\quad \left. + \left( \lambda_3^2 - \frac{\lambda_1^2 \lambda_3^1}{\lambda_1^1} \right) A_3 r^1 (1 + B_3 (r^1)^2)^{-1/2} \right] t - \vec{\lambda}^2 \cdot \vec{x}.
 \end{aligned} \tag{6.38}$$

Case  $(E_1 E_2 S_3)$ . The nonscattering rank-2 solution  $(E_1^+ E_2^+ S_3)$  invariant under the vector field

$$X = \frac{\partial}{\partial x^3} - \frac{\epsilon_{ijk} e_i^1 e_j^2 (\vec{e}^3 \times \vec{m}^3)_k}{\beta_{12}} \frac{\partial}{\partial t} + \frac{\beta_{23}}{\beta_{12}} \frac{\partial}{\partial x^1} + \frac{\beta_{31}}{\beta_{12}} \frac{\partial}{\partial x^2}, \tag{6.39}$$

with

$$\begin{aligned}
 \beta_{ij} &= (e_j^1 e_i^2 - e_i^1 e_j^2) [\vec{u}, \vec{e}^3, \vec{m}^3] + (e_j^2 (\vec{e}^3 \times \vec{m}^3)_i - e_i^2 (\vec{e}^3 \times \vec{m}^3)_j) (a + \vec{e}^1 \cdot \vec{u}), \\
 &\quad + (e_i^1 (\vec{e}^3 \times \vec{m}^3)_j - e_j^1 (\vec{e}^3 \times \vec{m}^3)_i) (a + \vec{e}^2 \cdot \vec{u}), \quad i, j = 1, 2, 3,
 \end{aligned} \tag{6.40}$$

has the form

$$\begin{aligned}
 a &= \frac{A_1 ((e_1^1 + e_1^2)x^1 + (e_2^1 + e_2^2)x^2)}{1 - A_1(1 + \kappa)t}, \quad u^3 = u_0^3, \\
 u^1 &= \frac{-\kappa A_1 (((e_1^1)^2 + (e_1^2)^2)x^1 + (e_1^1 e_2^1 + e_1^2 e_2^1)x^2) - u_3^1(r^3)}{1 - A_1(1 + \kappa)t}, \\
 u^2 &= \kappa A_1 \left( \frac{e_2^1 (\beta u_3^1(r^3)t - e_1^1 x^1 - e_2^1 x^2)}{1 - A_1(1 + \kappa)t} \right. \\
 &\quad \left. + \frac{e_2^2 (-\beta u_3^1(r^3)t - e_1^2 x^1 - e_2^2 x^2)}{1 - A_1(1 + \kappa)t} \right) + \frac{e_2^2 - e_2^1}{e_1^2 - e_1^1} u_3^1(r^3),
 \end{aligned} \tag{6.41}$$

where  $|\vec{e}^1|^2 = |\vec{e}^2|^2 = 1, \vec{e}^1 \cdot \vec{e}^2 = -\kappa^{-1}, e_3^1 = e_3^2 = 0, \beta = (1 + \kappa^{-1}) / (e_1^1 - e_1^2)$  and  $A_1, u_0^3 \in \mathbb{R}$ . The Riemann invariants are

$$r^1 = \frac{\beta u_3^1(r^3)t - e_1^1 x^1 - e_2^1 x^2}{1 - A_1(1 + \kappa)t}, \quad r^2 = \frac{-\beta u_3^1(r^3)t - e_1^2 x^1 - e_2^2 x^2}{1 - A_1(1 + \kappa)t}, \quad r^3 = x^3 - u_0^3 t, \tag{6.42}$$

where  $u_3^1$  is an arbitrary function of  $r^3$ .

This solution represents a Riemann double wave. It does not admit removable singularities for any choice of  $u_3^1(r^3)$ , but the functions  $a, u^1$  and  $u^2$  are subject to the gradient catastrophe at the time  $T = (A_1(1 + \kappa))^{-1}$ .

Case  $(E_1 S_2 S_3)$ . The nonscattering rank-2 solution  $(E_1^+ S_2 S_3)$  invariant under the vector field

$$X = \frac{\partial}{\partial x^3} + \frac{\epsilon_{ijk} e_i^1 (\vec{e}^2 \times \vec{m}^2)_j (\vec{e}^3 \times \vec{m}^3)_k}{\beta_{12}} \frac{\partial}{\partial t} + \frac{\beta_{23}}{\beta_{12}} \frac{\partial}{\partial x^1} + \frac{\beta_{31}}{\beta_{12}} \frac{\partial}{\partial x^2}, \tag{6.43}$$

with

$$\begin{aligned}
 \beta_{ij} &= [(\vec{e}^2 \times \vec{m}^2)_i (\vec{e}^3 \times \vec{m}^3)_j - (\vec{e}^2 \times \vec{m}^2)_j (\vec{e}^3 \times \vec{m}^3)_i] (a + \vec{e}^1 \cdot \vec{u}) \\
 &\quad + [e_j^1 (\vec{e}^3 \times \vec{m}^3)_i - e_i^1 (\vec{e}^3 \times \vec{m}^3)_j] [\vec{u}, \vec{e}^2, \vec{m}^2] \\
 &\quad + [e_i^1 (\vec{e}^2 \times \vec{m}^2)_j - e_j^1 (\vec{e}^2 \times \vec{m}^2)_i] [\vec{u}, \vec{e}^3, \vec{m}^3],
 \end{aligned} \tag{6.44}$$

is given by

$$\begin{aligned}
 a &= A_1 \frac{(C_2/\lambda_1^2 + C_3/\lambda_1^3)t - x^1}{1 - A_1(1 + \kappa)t}, & u^1 &= \frac{(C_2/\lambda_1^2 + C_3/\lambda_1^3)(1 - A_1 t) - \kappa A_1 x^1}{1 - A_1(1 + \kappa)t}, \\
 u^2 &= C(b\lambda_1^2 - \lambda_1^3)(\lambda_2^2 x^2 + \lambda_3^2 x^3), & u^3 &= -\frac{C\lambda_2^2(b\lambda_1^2 - \lambda_1^3)}{\lambda_3^2 \lambda_1^3}(\lambda_2^2 x^2 + \lambda_3^2 x^3).
 \end{aligned}
 \tag{6.45}$$

The Riemann invariants have the explicit form

$$\begin{aligned}
 r^1 &= \frac{(C_2/\lambda_1^2 + C_3/\lambda_1^3)t - x^1}{1 - A_1(1 + \kappa)t}, & A_1, C &\in \mathbb{R}, \\
 r^2 &= \left( \kappa A_1 \frac{(C_2 + \lambda_1^2/\lambda_1^3)t - \lambda_1^2 x^1}{1 - A_1(1 + \kappa)t} + C_2 + \frac{\lambda_1^2}{\lambda_1^3} C_3 \right) t - \vec{\lambda}^2 \cdot \vec{x}, \\
 r^3 &= \left( \kappa A_1 \frac{(\lambda_1^3/\lambda_1^2 + C_3)t - \lambda_1^3 x^1}{1 - A_1(1 + \kappa)t} + \frac{\lambda_1^3}{\lambda_1^2} C_2 + C_3 \right) t - \lambda_1^3 x^1 - b(\lambda_2^2 x^2 + \lambda_3^2 x^3).
 \end{aligned}
 \tag{6.46}$$

Here, we introduced the notation  $\lambda^{S_i} = (\lambda_0^i, \vec{\lambda}^i)$ ,  $C_i = [\vec{u}_i, \vec{e}^i, \vec{m}^i]$ ,  $i = 2, 3$  and  $\vec{\lambda}^3 = (\lambda_1^3, b\lambda_2^2, b\lambda_3^2)$ ,  $b \in \mathbb{R}$ . Note that  $a$  and  $u^1$  both admit the gradient catastrophe at the time  $T = (A_1(1 + \kappa))^{-1}$  while  $u^2$  and  $u^3$  are stationary. In this case the solution again has a form of Riemann double wave.

Case  $(S_1 S_2 S_3)$ . The rank-2 solution is invariant under the vector field

$$X = \frac{\partial}{\partial t} + u^1 \frac{\partial}{\partial x^1} + u^2 \frac{\partial}{\partial x^2} + u^3 \frac{\partial}{\partial x^3}.
 \tag{6.47}$$

In this case subjecting the initial system (4.4) to the DCs (3.17) leads to the overdetermined system

$$a = a_0, \quad \vec{u} + (\vec{u} \cdot \nabla)\vec{u} = 0, \quad \nabla \vec{u} = 0, \quad a_0 \in \mathbb{R}.
 \tag{6.48}$$

The solution of (6.48) is divergence free if and only if

$$\vec{u} = f(r^1, r^2, r^3), \quad f : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad r^i = x^i - u^i t, \quad i = 1, 2, 3.
 \tag{6.49}$$

The Jacobi matrix  $Df(r) = (\partial f^\alpha / \partial r^i)$  has to be nilpotent. In fact, the reduced system (6.48) mandates that the characteristic polynomial is equal to

$$\det(\lambda \mathcal{I}_3 + Df(r)) = \lambda^3 - \lambda^2 \text{tr}(f_{,ri}^\alpha) + \frac{1}{2}[(\text{tr}(f_{,ri}^\alpha))^2 - \text{tr}(f_{,ri}^\alpha)^2] \lambda + \det(f_{,ri}^\alpha) = \lambda^3.
 \tag{6.50}$$

In order to satisfy this condition we can select the arbitrary functions  $f^\alpha$  in the following way  $f^1 = b(r^2, r^3)$  and  $f^2 = f^3 = g(r^2 - r^3)$ . Then we have

$$Df(r) = \begin{pmatrix} 0 & b_{r^2} & b_{r^3} \\ 0 & g_s & -g_s \\ 0 & g_s & -g_s \end{pmatrix}, \quad s = r^2 - r^3.
 \tag{6.51}$$

If  $b_{r^2} \neq b_{r^3}$ , then  $\text{rank } Df(r) = 2$ , otherwise  $f^1$  is an arbitrary function of one variable, i.e.  $f^1 = h(r^2 - r^3)$ , and  $\text{rank } Df(r) = 1$ . In the rank-2 case the solution has the form

$$\begin{aligned}
 u^1(x, t) &= b(x^2 - t g(x^2 - x^3), x^3 - t g(x^2 - x^3)), \\
 u^2(x, t) &= u^3(x, t) = g(x^2 - x^3), \quad a = a_0, \quad a_0 \in \mathbb{R},
 \end{aligned}
 \tag{6.52}$$

where  $b$  is an arbitrary function of two variables  $(x^2 - u^2 t)$  and  $(x^3 - u^3 t)$ , and  $g$  is an arbitrary function of  $(x^2 - x^3)$ .

Depending on the choice of the arbitrary functions, the relations (6.52) can lead to elementary solutions (constant, algebraic, with one or two poles, trigonometric, hyperbolic)



or doubly periodic solutions which can be expressed in terms of Jacobi’s elliptic functions  $\text{sn}$ ,  $\text{cn}$  and  $\text{dn}$ . To ensure that the elliptic solutions possess one real and one purely imaginary period and that, for real argument  $r^i$ , they are contained in the interval between  $-1$  and  $+1$ , the moduli  $k$  of the elliptic functions have to satisfy the condition  $0 < k^2 < 1$ . An example of such elliptic solution has been obtained by submitting the arbitrary functions  $b$  and  $g$  to the DC (6.1). It has the explicit form

$$\begin{aligned} u^1 &= A_1[1 + B_1 \text{sn}^2(\beta(x^2 + nx^3) - (n + 1)[A_2[1 + B_2 \text{sn}^2(\beta(x^2 - x^3), k)]^{-1/2}], k)]^{-1/2}, \\ u^2 = u^3 &= A_2[1 + B_2 \text{sn}^2(\beta(x^2 - x^3), k)]^{-1/2} \text{sn}(\beta(x^2 - x^3), k), \\ a &= a_0, \quad 0 < k^2 < 1, \quad A_i, B_i, \beta \in \mathbb{R}, \quad B_i > 0, \quad i = 1, 2. \end{aligned} \tag{6.53}$$

This is a bounded solution representing a snoidal double wave.

**7. Rank-3 solutions**

Let us now present the rank-3 solutions obtained by way of the procedure analogical to the one described in section 3 for the rank-2 solutions. Here, the only difference is that the DC of the Klein–Gordon form (6.1) include three independent variables:  $r^1, r^2, r^3$ .

Case  $(E_1 E_2 E_3)$ . The rank-3 potential solution  $(E_1^+ E_2^+ E_3^+)$  invariant under the vector field

$$X = \frac{\partial}{\partial x^3} - \frac{[\vec{e}^1, \vec{e}^2, \vec{e}^3]}{\beta_3} \frac{\partial}{\partial t} + \frac{\beta_1}{\beta_3} \frac{\partial}{\partial x^1} + \frac{\beta_2}{\beta_3} \frac{\partial}{\partial x^2}, \tag{7.1}$$

with  $\beta_i = (\vec{e}^2 \times \vec{e}^3)_i(a + \vec{e}^1 \cdot \vec{u}) + (\vec{e}^1 \times \vec{e}^3)_i(a + \vec{e}^2 \cdot \vec{u}) + (\vec{e}^1 \times \vec{e}^2)_i(a + \vec{e}^3 \cdot \vec{u})$ , takes the form  $a = a_1(r^1) + a_2(r^2) + a_3(r^3)$ ,  $\vec{u} = \kappa(\vec{e}^1 a_1(r^1) + \vec{e}^2 a_2(r^2) + \vec{e}^3 a_3(r^3))$ ,

where the Riemann invariants are

$$r^i(x, u) = (1 + \kappa)a_i(r^i)t - \vec{e}^i \cdot \vec{x}, \quad \vec{e}^i \cdot \vec{e}^j = -\kappa^{-1}, \quad |\vec{e}^i|^2 = 1, \quad i \neq j = 1, 2, 3, \tag{7.3}$$

and  $a_i$  are arbitrary functions of  $r^i$ . Note that, just as in the case  $E_1 E_2$ , the angle  $\varphi_{ij}$  between each pair of wave vectors  $\vec{e}^i, \vec{e}^j, i \neq j = 1, 2, 3$ , has to satisfy the condition (6.8). This nonscattering rank-3 solution coincides with the one obtained previously by the GMC [39].

- (i) After submitting the arbitrary function  $a_i$  to the DC (6.1) modified for the case of three dimensions, we obtain several bounded solutions. We list here two examples.

An interesting case is the algebraic kink solution

$$a = \sum_{i=1}^3 A_i r^i (1 + B_i (r^i)^2)^{-1/2}, \quad \vec{u} = \kappa \left[ \sum_{i=1}^3 A_i r^i (1 + B_i (r^i)^2)^{-1/2} \vec{e}^i \right], \tag{7.4}$$

where the Riemann invariants are given by

$$r^i = [(1 + \kappa)A_i r^i (1 + B_i (r^i)^2)^{-1/2}]t - \vec{e}^i \cdot \vec{x}, \quad A_i, B_i \in \mathbb{R} \quad i = 1, 2, 3. \tag{7.5}$$

This solution evolves as a triple wave and is bounded even when the Riemann invariants  $r^i$  admit the gradient catastrophe at the time  $T_i = (1 + \kappa)^{-1} A_i^{-1}$ .

- (ii) Another interesting solution describes an algebraic solitary triple wave of a kink type

$$a = \sum_{i=1}^3 A_i (1 + e^{B_i r^i})^{-1/2}, \quad \vec{u} = \kappa \sum_{i=1}^3 A_i (1 + e^{B_i r^i})^{-1/2} \vec{e}^i, \quad A_i, B_i \in \mathbb{R} \tag{7.6}$$

where the Riemann invariants are given by

$$r^i = [(1 + \kappa)A_i (1 + e^{B_i r^i})^{-1/2}]t - \vec{e}^i \cdot \vec{x}, \quad i = 1, 2, 3. \tag{7.7}$$

The Riemann invariants admit the gradient catastrophe at the time

$$T_i = -2^{5/2}((1 + \kappa)A_i B_i)^{-1}, \tag{7.8}$$

but the solution remains bounded. In both cases the angle  $\varphi_{ij}$  between the wave vectors  $\vec{e}^i$  and  $\vec{e}^j$  is given by (6.8).

Case  $(E_1 S_2 S_3)$ . In this case we have to distinguish two situations, depending on the choice of wave vectors  $\lambda^{E_1}$ ,  $\lambda^{S_2}$  and  $\lambda^{S_3}$ .

First we look for the rank-3 solution  $(E_1^+ S_2 S_3)$  invariant under the vector field

$$X = e_1^1 \frac{\partial}{\partial x^1} + e_2^2 \frac{\partial}{\partial x^2}, \tag{7.9}$$

where we have assumed that the linearly independent wave vectors associated with the waves  $E_1^+$ ,  $S_2$  and  $S_3$  are given by

$$\lambda^{E_1^+} = (a + u^3, 0, 0, -1), \quad \lambda^{S_2} = (e_2^2 u^1 - e_2^1 u^2, -e_2^2, e_2^1, 0), \quad \lambda^{S_3} = (-e_3^3 u^3, 0, 0, e_3^3). \tag{7.10}$$

The corresponding Riemann invariants satisfy the following relations

$$r^1 = ((1 + \kappa^{-1})f(r^1) + a_0 + u_0^3)t - x^3, \quad r^2 = t - x^1 \sin g(r^2, r^3) + x^2 \cos g(r^2, r^3),$$

where  $r^3$  obeys the evolutionary partial differential equation

$$\frac{\partial r^3}{\partial t} + (f(r^1) + u_0^3) \frac{\partial r^3}{\partial x^3} = 0. \tag{7.11}$$

The solution then takes the form

$$\begin{aligned} a &= \kappa^{-1} f(r^1) + a_0, & u^1 &= \sin g(r^2, r^3), \\ u^2 &= -\cos g(r^2, r^3), & u^3 &= f(r^1) + u_0^3, \end{aligned} \quad a_0, u_0^3 \in \mathbb{R}, \tag{7.12}$$

where  $f$  is an arbitrary function of  $r^1$  and  $g$  is an arbitrary function of  $r^2$  and  $r^3$ . This scattering rank-3 solution has been obtained earlier through the GMC [39].

(i) If  $f(r^1) = A_1 r^1 + B_1$ , then the solution of (7.11) can be integrated in a closed form

$$\begin{aligned} a &= \kappa^{-1}(A_1 r^1 + B_1) + a_0, & u^1 &= \sin g(r^2, r^3), \\ u^2 &= -\cos g(r^2, r^3), & u^3 &= A_1 r^1 + B_1 + u_0^3, \end{aligned} \quad A_1, B_1 \in \mathbb{R}, \tag{7.13}$$

and the Riemann invariants are given by

$$\begin{aligned} r^1 &= \frac{((1 + \kappa^{-1})B_1 + a_0 + u_0^3)t - x^3}{1 - (1 + \kappa^{-1})A_1 t}, \\ r^2 &= t - x^1 \sin g(r^2, r^3) + x^2 \cos g(r^2, r^3), \\ r^3 &= \Psi \left[ \frac{1}{A_1} (A_1(\kappa a_0 - u_0^3)t + x^3 - \kappa a_0 - B_1)((1 + \kappa)A_1 t - \kappa)^{-\kappa/\kappa+1} \right], \end{aligned} \tag{7.14}$$

where  $\Psi$  is an arbitrary function of its argument and  $g$  is an arbitrary function of two variables  $r^2$  and  $r^3$ . This solution corresponds to a scattering Riemann triple wave.

After subjecting the arbitrary functions  $f$  and  $g$ , appearing in the solution (7.12), to the modified DC (6.1) we get several bounded solutions. Below, we present two of them.

(ii) A physically interesting subcase of  $(E_1^+ S_2 S_3)$  is the solution

$$\begin{aligned} a &= \kappa^{-1} A_1 [1 + B_1 (1 + \cosh(C_1 r^1))]^{-1/2} + a_0, \\ u^1 &= \sin \left[ \frac{A_2(R)^{-1/2} \tan y}{(B_2 + \tan^2 y)^{1/2}} \right], & A_i, B_i, C_1 &\in \mathbb{R}, & B_i > 0, & i = 1, 2, \\ u^2 &= -\cos \left[ \frac{A_2(R)^{-1/2} \tan y}{(B_2 + \tan^2 y)^{1/2}} \right], \\ u^3 &= A_1 [1 + B_1 (1 + \cosh(C_1 r^1))]^{-1/2} + u_0^3. \end{aligned} \tag{7.15}$$

Here we introduced the following notation  $R = (r^2)^2 + (r^3)^2$  and  $y = \frac{1}{2} \ln|D_1 R|$ , with  $D_1 \in \mathbb{R}$ . The Riemann invariants  $r^1$  and  $r^2$  are

$$\begin{aligned} r^1 &= (1 + \kappa^{-1})A_1[1 + B_1(1 + \cosh(C_1 r^1))]^{-1/2}t - x^3, \\ r^2 &= t - x^1 \sin \left[ \frac{A_2(R)^{-1/2} \tan y}{(B_2 + \tan^2 y)^{1/2}} \right] + x^2 \cos \left[ \frac{A_2(R)^{-1/2} \tan y}{(B_2 + \tan^2 y)^{1/2}} \right] \end{aligned} \tag{7.16}$$

and  $r^3$  satisfies the linear partial differential equation

$$\frac{\partial r^3}{\partial t} + (A_1[1 + B_1(1 + \cosh(C_1 r^1))]^{-1/2} + u_0^3) \frac{\partial r^3}{\partial x^3} = 0. \tag{7.17}$$

This solution is finite everywhere except at  $R = 0$ , but has discontinuities for  $\ln|D_1 R| = (2n + 1)\pi, n \in \mathbb{Z}$ . It remains bounded even when the Riemann invariants  $r^1, r^2$  and  $r^3$  tend to infinity. Physically, this solution represents nonstationary concentric waves damped by the factor  $R^{-1/2}$ .

(iii) Another solution worth mentioning has the form of an algebraic solitary wave

$$\begin{aligned} a &= \kappa^{-1}A_1[1 + B_1(1 + \cosh C_1 r^1)]^{-1/2} + a_0, \quad B_1, C_1 > 0, \\ u^1 &= \sin(D_1[1 + e^{h(r^2, r^3)}]^{-1/2}), \quad A_1, B_1, C_1, D_1 \in \mathbb{R}, \\ u^2 &= \cos(D_1[1 + e^{h(r^2, r^3)}]^{-1/2}), \\ u^3 &= A_1[1 + B_1(1 + \cosh C_1 r^1)]^{-1/2} + u_0^3, \end{aligned} \tag{7.18}$$

where  $h$  is an arbitrary function of  $r^2$  and  $r^3$ . The Riemann invariants are

$$\begin{aligned} r^1 &= (1 + \kappa^{-1})A_1[1 + B_1(1 + \cosh(C_1 r^1))]^{-1/2}t - x^3, \\ r^2 &= t - x^1 \sin D_1[1 + e^{h(r^2, r^3)}]^{-1/2} + x^2 \cos D_1[1 + e^{h(r^2, r^3)}]^{-1/2}, \end{aligned} \tag{7.19}$$

and  $r^3$  satisfies the partial differential equation (7.17).

We now consider the case  $(E_1^+ S_2 S_3)$  with a different selection of the wave vectors than that assumed in (7.10), namely we choose

$$\begin{aligned} \lambda^{E_1^+} &= (a + e_1^1 u^1 + e_1^2 u^2, -e_1^1, -e_1^2, 0), \quad |e_1|^2 = 1, \\ \lambda^{S_2} &= (u^2, 0, -1, 0), \quad \lambda^{S_3} = (-u^1, 1, 0, 0). \end{aligned} \tag{7.20}$$

This leads to scattering rank-3 solution of the form

$$\begin{aligned} a &= \kappa^{-1}f(r^1) + a_0, \quad u^1 = \sin f(r^1), \quad u^2 = -\cos f(r^1), \\ u^3 &= g(r^2 \cos f(r^1) + r^3 \sin f(r^1)), \quad a_0 \in \mathbb{R}, \end{aligned} \tag{7.21}$$

in which  $g$  is an arbitrary function of one variable  $r^2 \cos f(r^1) + r^3 \sin f(r^1)$ . The Riemann invariants are

$$\begin{aligned} r^1 &= (\kappa^{-1}f(r^1) + a_0)t - x^1 \cos f(r^1) - x^2 \sin f(r^1), \\ r^2 &= -t \cos f(r^1) - x^2, \quad r^3 = -t \sin f(r^1) + x^1. \end{aligned} \tag{7.22}$$

This triple wave solution coincides with the one obtained through the GMC [39].

(iv) As previously, we constructed particular solutions from (7.21) by requiring that the arbitrary function  $f$  satisfies the modified DC (6.1). One of the interesting examples is a periodic solution

$$\begin{aligned} a &= \kappa^{-1}A_1(1 - B_1 \cos C_1 r^1)^{-1/2} + a_0, \\ u^1 &= \sin A_1(1 - B_1 \cos C_1 r^1)^{-1/2}, \quad A_1, B_1, C_1 \in \mathbb{R}, \\ u^2 &= -\cos A_1(1 - B_1 \cos C_1 r^1)^{-1/2}, \quad |B_1| < 1, \\ u^3 &= g(r^2 \cos A_1(1 - B_1 \cos C_1 r^1)^{-1/2} + r^3 \sin A_1(1 - B_1 \cos C_1 r^1)^{-1/2}), \end{aligned} \tag{7.23}$$

with the Riemann invariants

$$\begin{aligned} r^1 &= (\kappa^{-1} A_1(1 - B_1 \cos C_1 r^1)^{-1/2} + a_0)t - x^1 \cos A_1(1 - B_1 \cos C_1 r^1)^{-1/2} \\ &\quad - x^2 \sin A_1(1 - B_1 \cos C_1 r^1)^{-1/2}, \\ r^2 &= -t \cos A_1(1 - B_1 \cos C_1 r^1)^{-1/2} - x^2, \\ r^3 &= -t \sin A_1(1 - B_1 \cos C_1 r^1)^{-1/2} + x^1. \end{aligned} \tag{7.24}$$

This solution remains bounded even when the Riemann invariants admit a gradient catastrophe.

### 8. Rank- $k$ solutions of fluid dynamics equations

Let us now consider the isentropic flow of an ideal and compressible fluid in the case when the sound velocity depends on time  $t$  only. System (4.4) in  $(k + 1)$  dimensions becomes

$$\begin{aligned} u_t + (u \cdot \nabla)u &= 0, \\ a_t + \kappa^{-1}a \operatorname{div} u &= 0, \quad a_{x^j} = 0, \quad j = 1, \dots, k, \quad a > 0, \quad \kappa = 2(\gamma - 1)^{-1}. \end{aligned} \tag{8.1}$$

We show that in this case our approach enables us to construct arbitrary rank solutions.

The change of coordinates on  $\mathbb{R}^{k+1} \times \mathbb{R}^{k+1}$

$$\bar{t} = t, \quad \bar{x}^1 = x^1 - u^1 t, \dots, \bar{x}^k = x^k - u^k t, \quad \bar{a} = a, \quad \bar{u} = u \in \mathbb{R}^k, \tag{8.2}$$

transforms (8.1) into the system

$$\frac{\partial \bar{u}}{\partial \bar{t}} = 0, \quad \frac{\partial \bar{a}}{\partial \bar{t}} + \kappa^{-1} \bar{a} \operatorname{tr}((\mathcal{I}_k + \bar{t} D\bar{u}(\bar{x}))^{-1} D\bar{u}(\bar{x})) = 0, \quad \frac{\partial \bar{a}}{\partial \bar{x}} = 0, \tag{8.3}$$

where  $D\bar{u}(\bar{x}) = \partial \bar{u} / \partial \bar{x} \in \mathbb{R}^{k \times k}$  is the Jacobian matrix and  $\bar{x} = (\bar{x}^1, \dots, \bar{x}^k) \in \mathbb{R}^k$ . The general solution of the conditions  $\partial \bar{u} / \partial \bar{t} = 0$  and  $\partial \bar{a} / \partial \bar{x} = 0$  is

$$\bar{u}(\bar{t}, \bar{x}) = f(\bar{x}), \quad \bar{a}(\bar{t}, \bar{x}) = \bar{a}(\bar{t}) > 0 \tag{8.4}$$

for any functions  $f : \mathbb{R}^k \rightarrow \mathbb{R}^k$  and  $\bar{a} : \mathbb{R} \rightarrow \mathbb{R}$ , respectively. Making use of (8.4) and of the trace identity

$$\frac{\partial}{\partial \bar{t}} (\ln \det B) = \operatorname{tr} \left( B^{-1} \frac{\partial B}{\partial \bar{t}} \right), \tag{8.5}$$

where  $B = (\mathcal{I}_k + \bar{t} Df(\bar{x}))$  and  $Df(\bar{x}) = \frac{\partial}{\partial \bar{t}} (\mathcal{I}_k + \bar{t} Df(\bar{x}))$ , we obtain from (8.3)

$$\frac{\partial}{\partial \bar{t}} [\ln(|\bar{a}(\bar{t})|^\kappa \det(\mathcal{I}_k + \bar{t} Df(\bar{x})))] = 0. \tag{8.6}$$

Differentiating (8.6) with respect to  $\bar{x}$  gives the condition on the flow velocity  $f(\bar{x})$

$$\frac{\partial^2}{\partial \bar{x} \partial \bar{t}} [\ln(\det(\mathcal{I}_k + \bar{t} Df(\bar{x})))] = 0. \tag{8.7}$$

Consequently, we have

$$\det(\mathcal{I}_k + \bar{t} Df(\bar{x})) = \alpha(\bar{x}) \beta(\bar{t}), \tag{8.8}$$

where  $\alpha$  and  $\beta$  are arbitrary functions of their argument. Evaluating (8.8) at  $\bar{t} = 0$  implies  $\alpha(\bar{x}) = \beta(0)^{-1}$ . Therefore,

$$\det(\mathcal{I}_k + \bar{t} Df(\bar{x})) = \frac{\beta(\bar{t})}{\beta(0)},$$

and we obtain

$$\frac{\partial}{\partial \bar{x}} \det(\mathcal{I}_k + \bar{t} Df(\bar{x})) = 0. \tag{8.9}$$

Equation (8.9) holds if and only if the coefficients  $p_n$ ,  $n = 0, \dots, k - 1$ , of the characteristic polynomial of the matrix  $Df(\bar{x})$  are constant. Thus the general solution of (8.1) is

$$\bar{u}(\bar{t}, \bar{x}) = f(\bar{x}), \quad \bar{a}(\bar{t}) = A_1(1 + p_{k-1}\bar{t} + \dots + p_0\bar{t}^k)^{-1/\kappa}, \quad A_1 \in \mathbb{R}^+, \quad (8.10)$$

with the Cauchy data

$$t = 0, \quad u(0, x) = f(x), \quad a(0) = A_1. \quad (8.11)$$

In the original coordinates  $(x, u) \in \mathbb{R}^p \times \mathbb{R}^q$  this rank- $k$  solution takes the form

$$u = f(x^1 - u^1 t, \dots, x^k - u^k t), \quad a(t) = A_1(1 + p_{k-1}t + \dots + p_0 t^k)^{-1/\kappa}. \quad (8.12)$$

Note that the sound velocity  $a$  is constant if and only if the Jacobian matrix  $Df(\bar{x})$  is nilpotent, i.e.

$$\det(-\lambda I_k + Df(\bar{x})) = (-\lambda)^k.$$

As an example let us consider the particular solution of (8.1) for  $k = 2$ . It is invariant under the vector fields

$$X_j = \frac{\partial}{\partial t} + u^j \frac{\partial}{\partial x^{(j)}}, \quad j = 1, 2.$$

The requirement that the coefficients  $p_n$  of the characteristic polynomial (3.64) of the Jacobi matrix  $Df(\bar{x})$  are constant means that

$$\det(D(f(\bar{x}))) = B_1, \quad \text{tr}(Df(\bar{x})) = 2C_1, \quad B_1, C_1 \in \mathbb{R},$$

where we denote  $B_1 = p_0$  and  $2C_1 = p_1$ . Solving the above conditions gives us the general rank-3 solution of (8.1) which is implicitly defined by

$$\begin{aligned} u^1(t, x, y) &= C_1(x - u^1 t) + \frac{\partial h}{\partial r^1}(x - u^1 t, y - u^2 t), \\ u^2(t, x, y) &= C_1(y - u^2 t) - \frac{\partial h}{\partial r^2}(x - u^1 t, y - u^2 t), \\ a(t) &= A_1((1 + C_1 t)^2 + B_1 t^2)^{-1/\kappa}, \quad A_1 \in \mathbb{R}^+, \end{aligned} \quad (8.13)$$

where the function  $h$  depends on two variables  $r^1 = x - u^1 t$  and  $r^2 = y - u^2 t$  and satisfies the nonhomogeneous Monge-Ampère equation

$$h_{r^1 r^1} h_{r^2 r^2} - h_{r^1 r^2} = B_1. \quad (8.14)$$

Depending on the selection of particular solutions of this equation we obtain Riemann double waves or other types of rank-2 solutions of (8.1).

## 9. Summary remarks

The objective of this paper was to develop a new systematic way of constructing rank- $k$  solutions of quasilinear hyperbolic systems of first order PDEs in many dimensions. Specifically, we have been interested in nonlinear superpositions of Riemann waves, which constitute the elementary solutions of these systems and are ubiquitous in the equations of mathematical physics. Interactions of Riemann waves are obviously present in many nonlinear physical phenomena. However there are still only a few examples of multiple rank solutions describing them in multi-dimensional systems. Most of these solutions were obtained through the generalized method of characteristics. The main idea behind our approach has been to look at this type of solutions from a different point of view, namely, to reformulate them in terms of symmetry group theory.

Let us now recapitulate our analysis. We look for rank- $k$  solutions of the system (3.1), expressible in terms of Riemann invariants,  $u = f(r^1(x, u), \dots, r^k(x, u))$ , where  $f: \mathbb{R}^k \rightarrow \mathbb{R}^q$ . Each Riemann invariant is associated with a specific wave vector involved in the interaction, i.e.  $r^A(x, u) = \lambda_i^A(u)x^i$ ,  $A = 1, \dots, k$ , where  $\ker(\lambda_i^A A^i(u)) \neq 0$ . The basic feature of these solutions is that they remain constant on  $(p - k)$ -dimensional hyperplanes perpendicular to the set of linearly independent wave vectors  $\lambda^1, \dots, \lambda^k$ . In the context of group theory, this means that the graph  $\{x, u(x)\}$  of these solutions is invariant under all vector fields  $X_a = \xi_a^i(u)\partial_{x^i}$  with  $\lambda_i^A \xi_a^i = 0$  for  $1 \leq a \leq p - k$ . Then  $u(x)$  is the solution of (3.1) for some function  $f$ , because the set  $\{r^1, \dots, r^k, u^1, \dots, u^q\}$  constitutes a complete set of invariants of the Abelian algebra  $L$  of such vector fields. The implicit form of these solutions leads to major difficulties in applying the classical symmetry reduction method to this case. To overcome these difficulties, we rectify the set of vector fields  $X_a$  by a change of variables on  $X \times U$ , choosing Riemann invariants as new independent variables. The initial equations (3.1) expressed in the new coordinates, complemented by the invariance conditions for the rectified vector fields  $X_a$ , form an overdetermined quasilinear system (3.21). Thus the solutions of this system are invariant under the Abelian group corresponding to  $L$ . The vector fields  $X_a$  constitute the conditional symmetries of the initial system (3.1). The consistency conditions for the overdetermined system (3.21), that is, the necessary and sufficient conditions for the existence of conditionally invariant solutions of (3.1), have been derived here and they take the form of the trace conditions (3.30) and (3.34). Given these conditions, we were able to devise a specific procedure for constructing solutions in terms of Riemann invariants. We present it for the case of rank-2 solutions, however higher rank solutions can, in principle, be constructed by analogy. The computational difficulties should not be underestimated here and in many cases additional assumptions are needed in order to perform integration or to arrive at compact forms of these solutions. Nevertheless, the implementation of the proposed CSM is still easier than that of the GMC. The latter imposes stronger restrictions on the wave vectors  $\lambda^A$ , which contribute to computational complexity as well as narrowing of the range of obtained solutions.

As the application to the isentropic flow equations shows, our approach has proved quite productive. We were able to reconstruct the general rank-2 and rank-3 solutions obtained via the GMC and to deliver several new classes of solutions, namely in the cases  $E_1 S_1$ ,  $S_1 S_2$ ,  $E_1 E_2 S_1$ ,  $E_1 S_1 S_2$  and  $S_1 S_2 S_3$ . For the equations of an isentropic flow with a sound velocity depending on time only (an assumption which simplifies things considerably) we obtained the arbitrary rank solution, together with the Cauchy conditions in a closed form.

Moreover, we present a simple technique which allows us to overcome the main weakness of solutions expressible in terms of Riemann invariants, resulting from the fact that the first derivatives of Riemann invariants, in most cases, tend to infinity after some finite time. We show that a proper selection of the arbitrary functions appearing in the general solution can lead to bounded solutions even in the cases when Riemann invariants admit a gradient catastrophe. We obtained numerous such solutions which, to our knowledge, are all new (we include here only some of them, namely (6.10), (6.22), (6.37), (6.53), (7.4), (7.6), (7.15), (7.18) and (7.23)). Most of these solutions have a soliton-like form and this fact is of note since the integrability properties of soliton theories do not easily generalize to more than two dimensions.

Our technique is applicable to a very wide class of systems, which includes many physically meaningful models. Given the promising results obtained, we expect it may be useful in such areas as nonlinear field equations, general relativity or equations of continuous media. Let us note also that, though the notion of Riemann invariants was originally defined for hyperbolic systems only, it seems that it can be easily adapted to elliptic systems. Since the conditional symmetry method can be applied to these systems, it is worth investigating

whether our approach to constructing rank- $k$  solutions can be extended to the elliptic case. Some preliminary analysis suggests that to be feasible.

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